Four heads are better than three

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Quick definitions/lies

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A *subshift* is a "language" whose "words" are colorings of the group, i.e. $X \subset \Sigma^G$ where G is the group, Σ a finite alphabet. (+ there are some axioms)

The head hierarchy

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Does this depend on algebraic and geometric properties of the group G?

It depends on torsion!

Theorem (S.-Törmä)

- *If G is a torsion group, the hierarchy does not collapse:*

$$\forall k \in \mathbb{N} : \mathcal{S}_{G,k} \neq \bigcup \mathcal{S}_{G,i}$$

- *If G is not a torsion group, the hierarchy collapses:*
  - *if G has decidable word problem*

$$\mathcal{S}_{G,3} = \bigcup \mathcal{S}_{G,i},$$

  - *and in general*

$$\mathcal{S}_{G,4} = \bigcup \mathcal{S}_{G,i}.$$

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If there exists a torsion group G where "long identities cannot be determined from short ones", then there exists a non-torsion group G' such that $\mathcal{S}_{G',3} \subsetneq \bigcup \mathcal{S}_{G',i}$.

The new contribution

Theorem (S.)

There exists a torsion group G where "long identities cannot be determined from short ones", thus there exists a non-torsion group G such that $\mathcal{S}_{G,3} \subsetneq \mathcal{S}_{G,4} = \bigcup \mathcal{S}_{G,i}$.

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Proof strategy: Abstract the problem away!

1. Define a recursion theoretic property for $A \subset \mathbb{N}$, which means roughly "knowing a prefix $A \upharpoonright n$ does not give you information about $m \in A$ for m large". Prove these r.e. sets like this exist.

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2. Prove that for any recursively enumerable $A \subset \mathbb{N}$ there exists a recursively-presented (bounded-)torsion group whose word problem is equivalent to A.
3. The solution is "a subshift defined by finite-state automata on a group of finite-state automata on a subshift on a group defined by finite-state automata"

Groups

A *group* G is a set together with an associative operation $\cdot : G^2 \to G$ (we write $gh = g \cdot h$), and admitting an identity $eg = ge = g$ and inverses $\forall g \in G : \exists g^{-1} \in G : gg^{-1} = g^{-1}g = e$.

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Write $\langle S \rangle$ for the smallest subgroup of G containing S. Our groups are always *finitely generated (f.g.)*: there is a finite set $S \Subset G$ with $G = \langle S \rangle$. We fix a symmetric generating set $S = \{s^{-1} \mid s \in S\}$ for every f.g. group.

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(In fact this more or less characterizes Cayley graphs.)

Example: $\mathbb{Z}^2$

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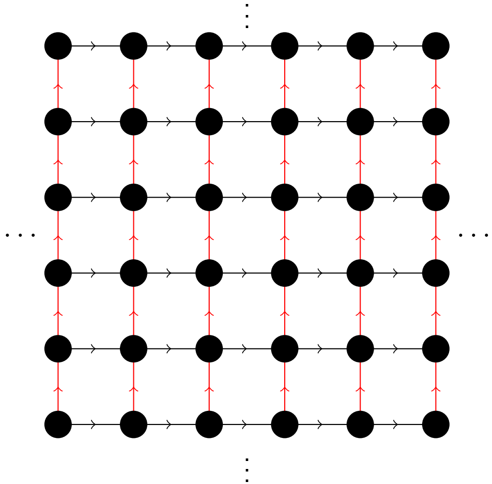
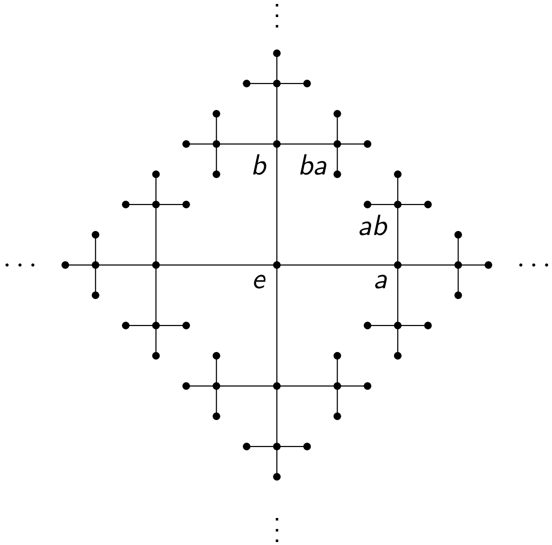


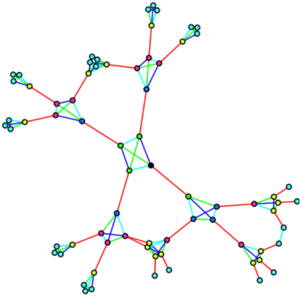
This is an abelian group: $\forall g, h : gh = hg$.

Example: free group $F_2 = \langle a, b \rangle$

$F_2 =$ 

Not abelian: $ab \neq ba$.

Example: Grigorchuk group



Generated by four involutions a, b, c, d ($aa = bb = cc = dd = e$). It is a torsion group: $\forall g \in G : \exists n > 0 : g^n = e$.

Example: Grigorchuk group definition

For completeness, the Grigorchuk group is

$$\langle a, b, c, d \rangle \leq \text{“the group of bijections on } \{0,1\}^{\omega}\text{”}$$

where on $\{0,1\}^{\mathbb{N}}$, for $\mathrm{x} \in \{0,1\}^{\omega}$ define

$$\begin{aligned} a \cdot 0\mathrm{x} &= 1\mathrm{x} & a \cdot 1\mathrm{x} &= 0\mathrm{x} \\ b \cdot 0\mathrm{x} &= 0(a \cdot \mathrm{x}) & b \cdot 1\mathrm{x} &= 1(c \cdot \mathrm{x}) \\ c \cdot 0\mathrm{x} &= 0(a \cdot \mathrm{x}) & c \cdot 1\mathrm{x} &= 1(d \cdot \mathrm{x}) \\ d \cdot 0\mathrm{x} &= 0\mathrm{x} & d \cdot 1\mathrm{x} &= 1(b \cdot \mathrm{x}) \end{aligned}$$

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Can be interpreted as a group of finite-state automata (acting on one-way infinite words).

Subshifts

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A *pattern* is $P \in \Sigma^D$ for some *domain* $D \Subset G$. If $(P_i)_{i \in \mathcal{I}}$ is a set of *forbidden patterns*, we say $x \in \Sigma^G$ avoids them if

$$\forall g \in G : \forall i \in \mathcal{I} : P_i \text{ does not appear at } g \text{ in } x,$$

where $P \in \Sigma^D$ *appears at g in x* if $x_{gd} = P_d$ for all $d \in D$.

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Standard exercise:

Theorem

A set $X \subset \Sigma^G$ is a subshift if and only if for some set of patterns $(P_i)_{i \in \mathcal{I}}$ it is the set of configurations $x \in \Sigma^G$ avoiding the patterns $(P_i)_{i \in \mathcal{I}}$.

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$\mathcal{S}_{G,k}$ is the class of subshifts you can define this way on the group G, using k automata.

The automaton model formally

A k-headed G-walking automaton is a tuple $\mathcal{A} = (Q, \Sigma, \delta, I, F)$ where $Q = Q_1 \times Q_2 \times \cdots Q_k$ are the (finite) *state sets*, Σ the *input alphabet*, $\delta = (\delta_1, \delta_2, \ldots, \delta_k)$ the *transition functions*, $I \subset Q$ the *initial states* and $F \subset Q$ the *final states*. Defining $Q' = (Q_1 \cup \{\bot\}) \times \cdots \times (Q_k \cup \{\bot\})$, we have $\delta_i : Q' \times \Sigma \to Q_i \times G$.

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An *instantaneous description (ID)* of such a machine is an element of $G^k \times Q$. A *run* of $\mathcal{A}$ on $x \in \Sigma^G$ is a sequence of IDs $(g_{t,1}, ..., g_{t,k}, q_{t,1}, \cdots q_{t,k})$, where $t = 0, 1, 2, ...$ such that

$$g_{0,1} = \cdots = g_{0,k} \wedge (q_{0,1}, \cdots, q_{0,k}) \in I,$$

and for all t and i,

$$\delta_i(q^i_{t,1}, \cdots, q^i_{t,k}, x_{g_{t,i}}) = (q_{t+1,i}, s_{t+1,i})$$

such that $g_{t+1,i} = g_{t,i} \cdot s_{t+1,i}$, where $q^i_{t,j} = q_{t,j}$ if $g_{t,i} = g_{t,j}$, and $q^i_{t,j} = \bot$ otherwise.

The automaton model formally (cont'd)

A run is *rejecting* if for some $T \in \mathbb{N}$ we have $g_{T,1} = \cdots = g_{T,k}$ and $(q_{T,1}, \cdots, q_{T,k}) \in F$. Otherwise it is *accepting*.

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For $X \subset \Sigma^G$, X is accepted by a k-headed automaton $\mathcal{A}$ if and only if

$$\forall x \in \Sigma^G : x \in X \iff \forall g \in G, q \in I : \text{the run with}$$
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Define $\mathcal{S}_{G,k}$ as the class of sets $X \subset \Sigma^G$ (for finite alphabets Σ) which are accepted by k-headed automata. These X are easily seen to be subshifts.

Impredictable sets

Write $\mathcal{T}$ for the total computable functions, and $\mathcal{P}$ for the partial computable functions.

Definition

For a function ϕ, a set $A \subset \mathbb{N}$ is ϕ-*impredictable* if

$$\exists \psi \in \mathcal{T} : \forall \chi \in \mathcal{P} : \exists^{\infty} p : \psi(p) \in A \iff \chi(p, A \restriction \phi(p)) \downarrow .$$

Intuition: "$A \restriction \phi(p)$ doesn't tell you whether $\psi(p) \in A$".

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Proof by standard recursion theory tricks.

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The *word problem* of G is

$$\mathrm{WP}(G) = \{w \in S^* \mid w = e \text{ as an element of } G\}.$$

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For every recursively enumerable set $A \subset \mathbb{N}$ there exists a recursively-presented torsion group G whose the word problem is enumeration equivalent to A.

Corollary

For any $\phi \in \mathcal{T}$ there exists a recursively presented torsion group G whose word problem is ϕ-impredictable.

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- To control the complexity of the word problem, pick a suitable subshift.
- Torsion is automatic for the new group, since finite-state automata must loop when walking on a torsion group.

Details of construction

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On $X_A \times H$, we have

- bijections $\hat{g}$ for $g \in G$ which act by shifting: $\hat{g}(x, h) = (g \cdot x, h)$ and
- for $b \in \{0,1\}$ and $h \in H$, "conditional bijections"
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Then $K = \langle \{\hat{g}, h_b \mid g \in G, b \in \{0,1\}, h \in h\} \rangle$ has the desired properties.

Four heads is better than three

Idea for existence of non-torsion group needing four heads:

- (Proved earlier with Ilkka:) if a three-headed automaton is run on a configuration of shape $K \times \mathbb{Z}$ which is (totally) periodic, then the heads stay at bounded K-distance from each other (as some function of p and number of states).

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- Impredictability means that a prefix of the word problem of K doesn't tell us whether large identities hold, so if we encode the word problem of K into periodic configurations, we get a contradiction if we assume three heads define it.
- Four heads can just read the whole word problem, as we showed with Törmä, so they can define the subshift.

Four heads is better than three (cont'd)

Some deets:

- Take $G' = K \times \mathbb{Z}$ where K is recursively presented, has "reasonable" torsion (e.g. bounded/polynomial), and with ϕ-impredictable word problem for "unreasonable" ϕ (e.g. $\phi = \exp \circ \exp \circ \exp \circ \exp \circ \exp$).

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- Take ψ from definition of impredictability, and let $B = \{n \mid \psi(n) \in \mathrm{WP}(K)\}$.
- For $p \in \mathbb{N}$ define $x^p \in \{0,1\}^{G'}$ by

$$x^p_{(g,n)} = 1 \iff n \equiv 0 \bmod p$$

and consider the smallest subshift Y_B containing the set $\{x^p \mid p \in B\}$.

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- By impredictability and the choice of $\psi \in \mathcal{T}$,

$$\forall \chi \in \mathcal{P} : \exists^\infty p : \psi(p) \in \mathrm{WP}(K) \iff \chi(p, K \upharpoonright \phi(p)) \downarrow,$$

a contradiction since
$\psi(p) \in \mathrm{WP}(K) \iff p \in B \iff x^p \in Y_B$.

Conjectures

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Everything is optimal (for f.g. infinite G):

- *If G is torsion, the hierarchy is strict:*

$$\forall k : \mathcal{S}_{G,k} \subsetneq \mathcal{S}_{G,k+1}.$$

- *If G is not torsion then*
  - *if G has decidable word problem,*

$$\mathcal{S}_{G,1} \subsetneq \mathcal{S}_{G,2} \subsetneq \mathcal{S}_{G,3} = \bigcup \mathcal{S}_{G,i},$$

  - *and if G has undecidable word problem,*

$$\mathcal{S}_{G,1} \subsetneq \mathcal{S}_{G,2} \subsetneq \mathcal{S}_{G,3} \subsetneq \mathcal{S}_{G,4} = \bigcup \mathcal{S}_{G,i}.$$

The End

Thank you for listening!