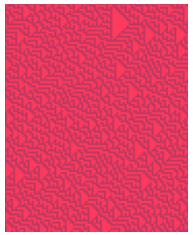
Topological Dynamical Properties in Turing Machines

Rodrigo Torres-Avilés[1]

1. Dpto. de Sistemas de Información, Universidad del Bío-Bío, Chile

AUTOMATA, August 10-12, 2020

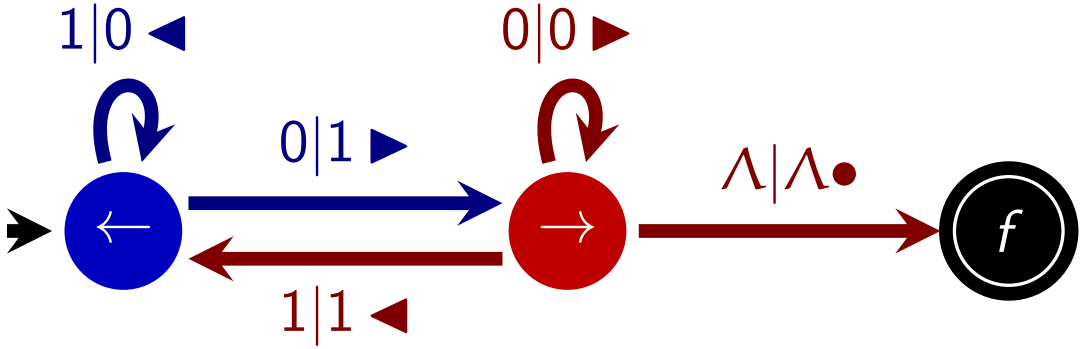


Content

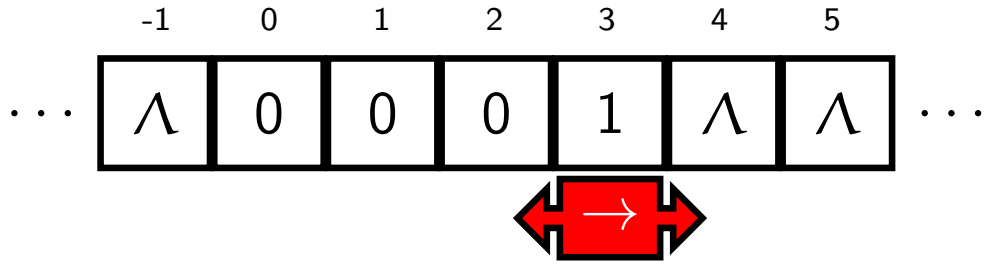
Traditional Turing Machine

Turing machine is a six-tuple: $T = (Q, \Sigma, q_0, F, \Lambda, \delta)$

Partial Transition function: $\delta : Q \times \Sigma \cup \{\Lambda\} \to Q \times \Sigma \cup \{\Lambda\} \times \{\blacktriangleleft, \bullet, \blacktriangleright\}$



Configuration: $(Q, \mathbb{Z}, (\Sigma \cup \{\Lambda\})^{\mathbb{Z}})$



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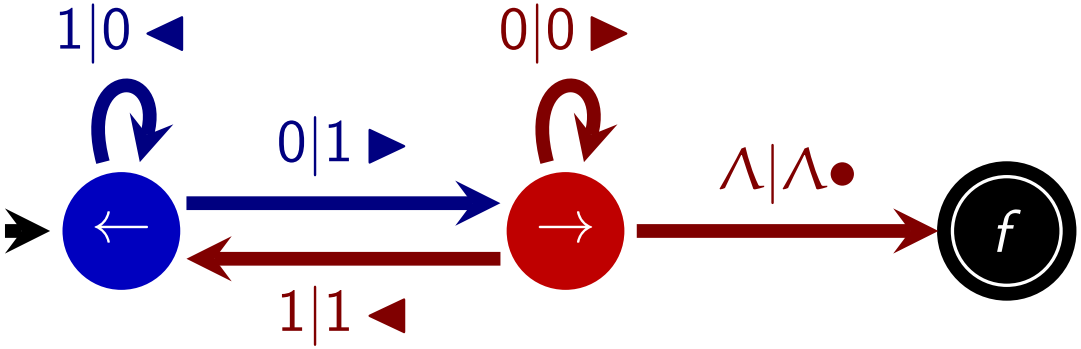

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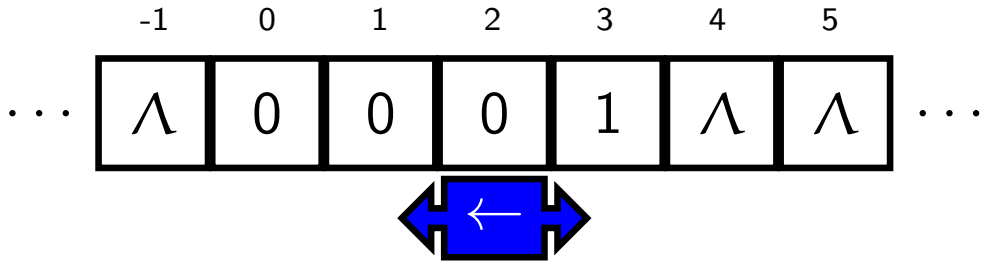

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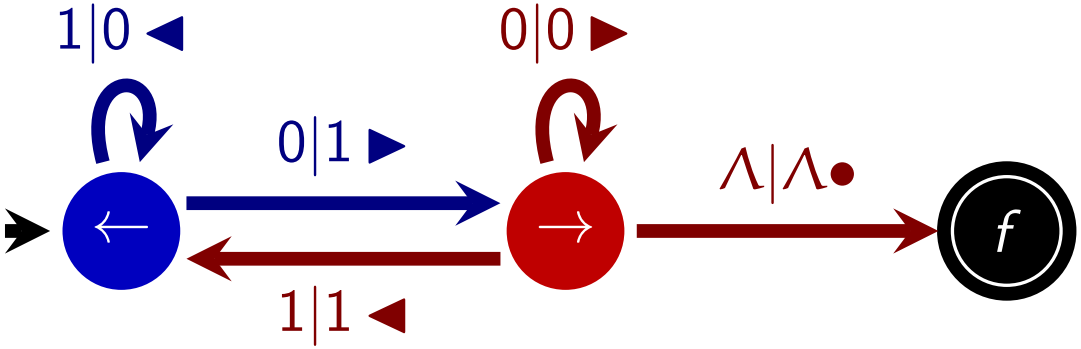

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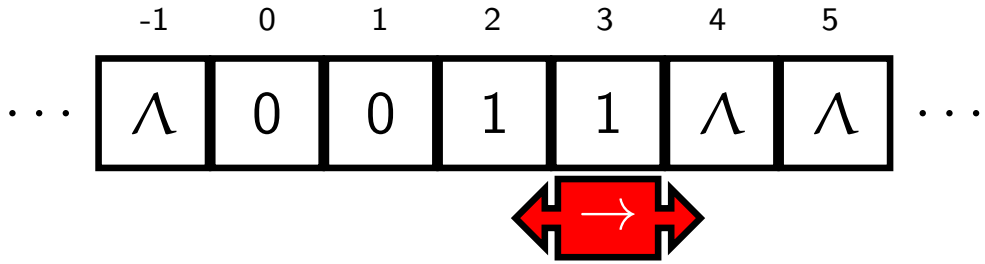
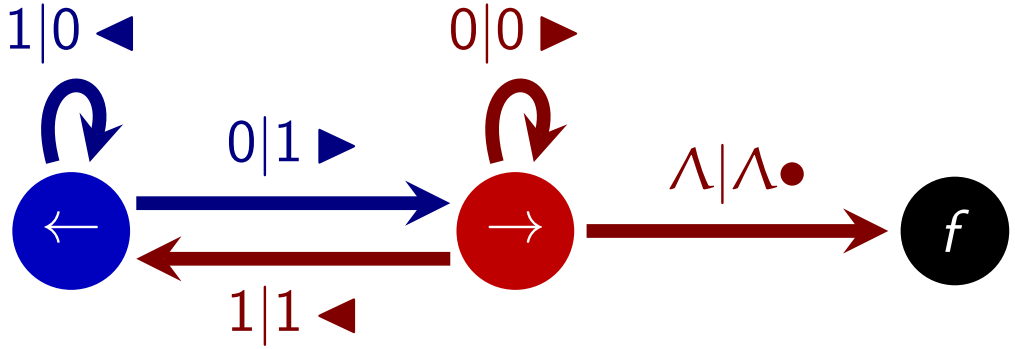


Converting to Turing Machine Dynamical Systems

Turing machine is a **four**-tuple: $T = (Q, \Sigma, \Lambda, \delta)$

Partial Transition function: $\delta : Q \times \Sigma \cup \{\Lambda\} \to Q \times \Sigma \cup \{\Lambda\} \times \{\blacktriangleleft, \bullet, \blacktriangleright\}$



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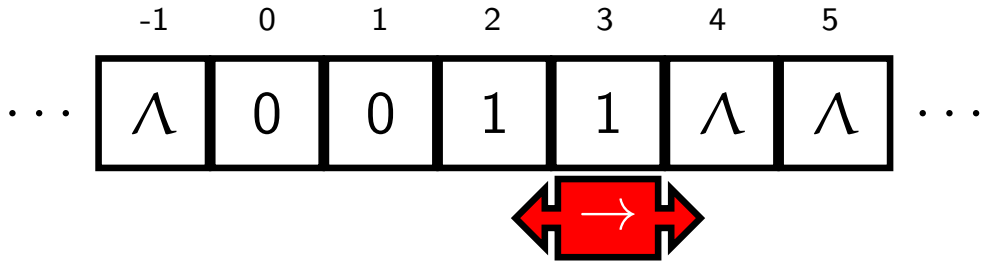
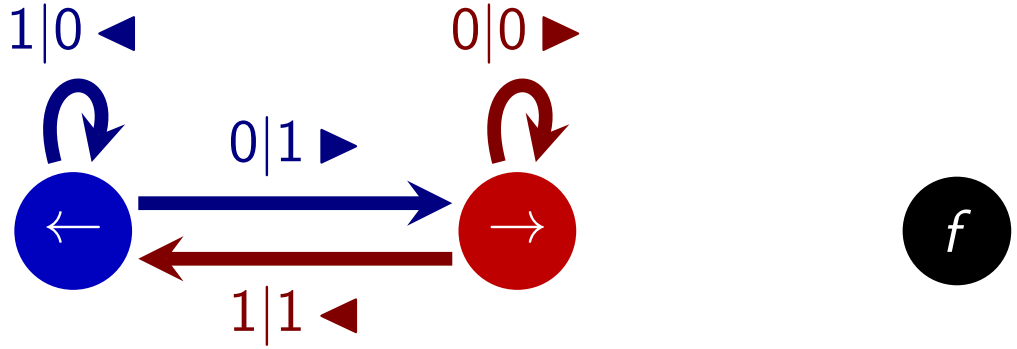


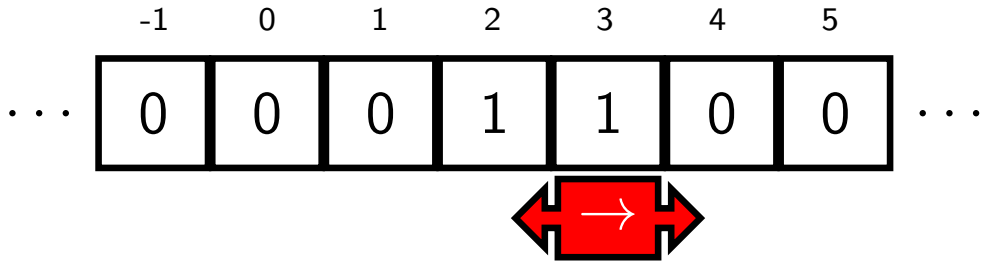
Converting to Turing Machine Dynamical Systems

Turing machine is a **three**-tuple: $T = (Q, \Sigma, \delta)$

Partial Transition function: $\delta : Q \times \Sigma \to Q \times \Sigma \times \{\blacktriangleleft, \bullet, \blacktriangleright\}$



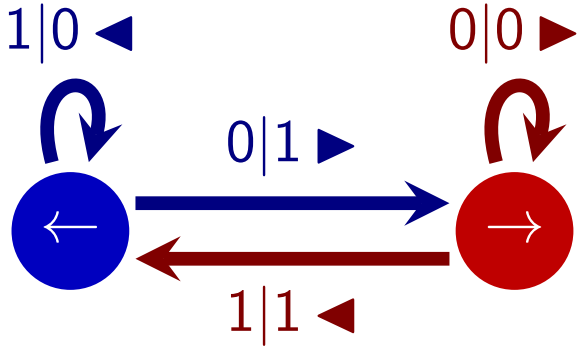
Configuration: $(Q, \mathbb{Z}, \mathbf{\Sigma}^{\mathbb{Z}})$



Converting to Turing Machine Dynamical Systems

Complete Turing machine: $T = (Q, \Sigma, \delta)$

Total Transition function: $\delta : Q \times \Sigma \to Q \times \Sigma \times \{\blacktriangleleft, \bullet, \blacktriangleright\}$



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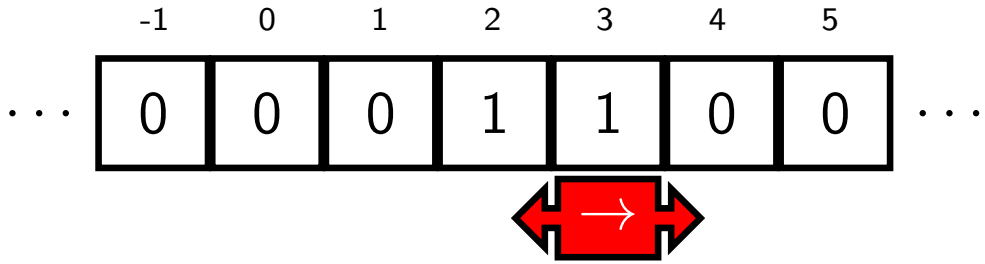

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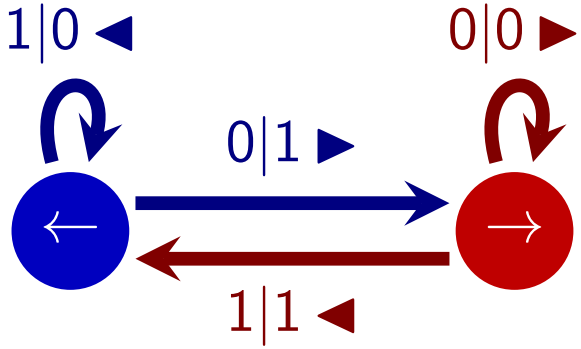
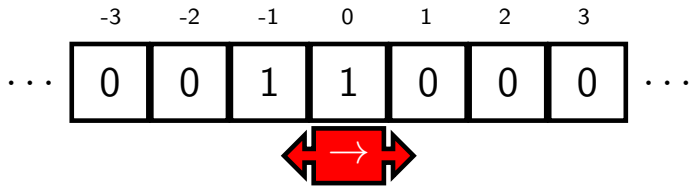
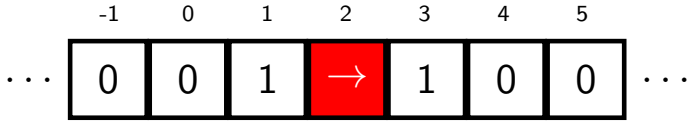


TMT Configuration: $({}^{\omega}\boldsymbol{\Sigma}, \boldsymbol{Q}, \boldsymbol{\Sigma}^{\omega})$



TMH Configuration: $\subset (\boldsymbol{\Sigma} \cup \boldsymbol{Q})^{\mathbb{Z}}$



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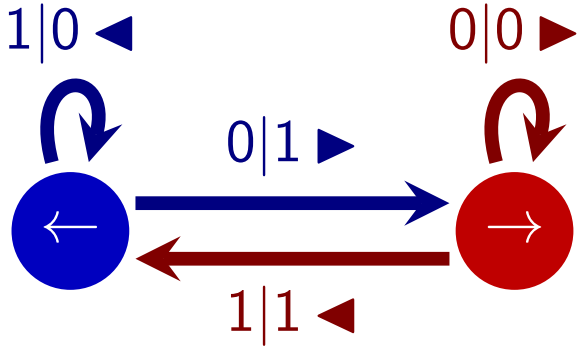

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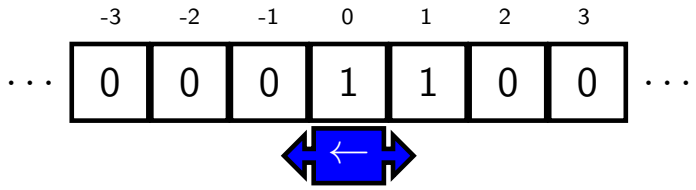

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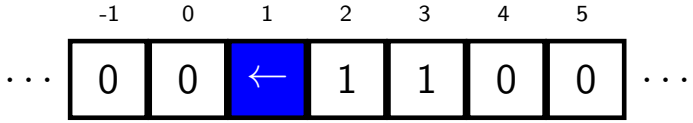

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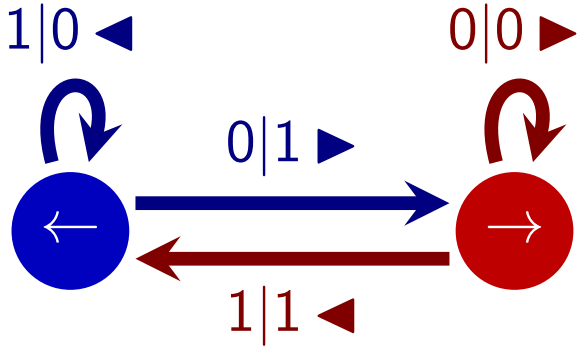

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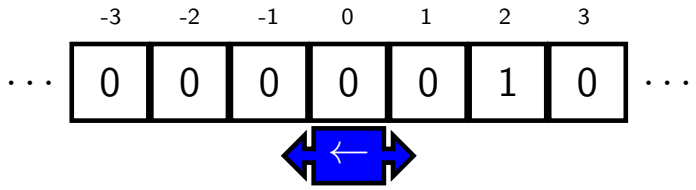

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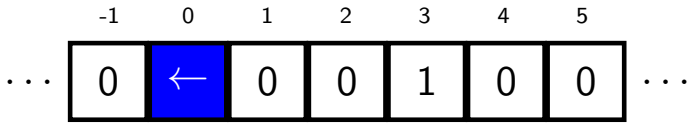
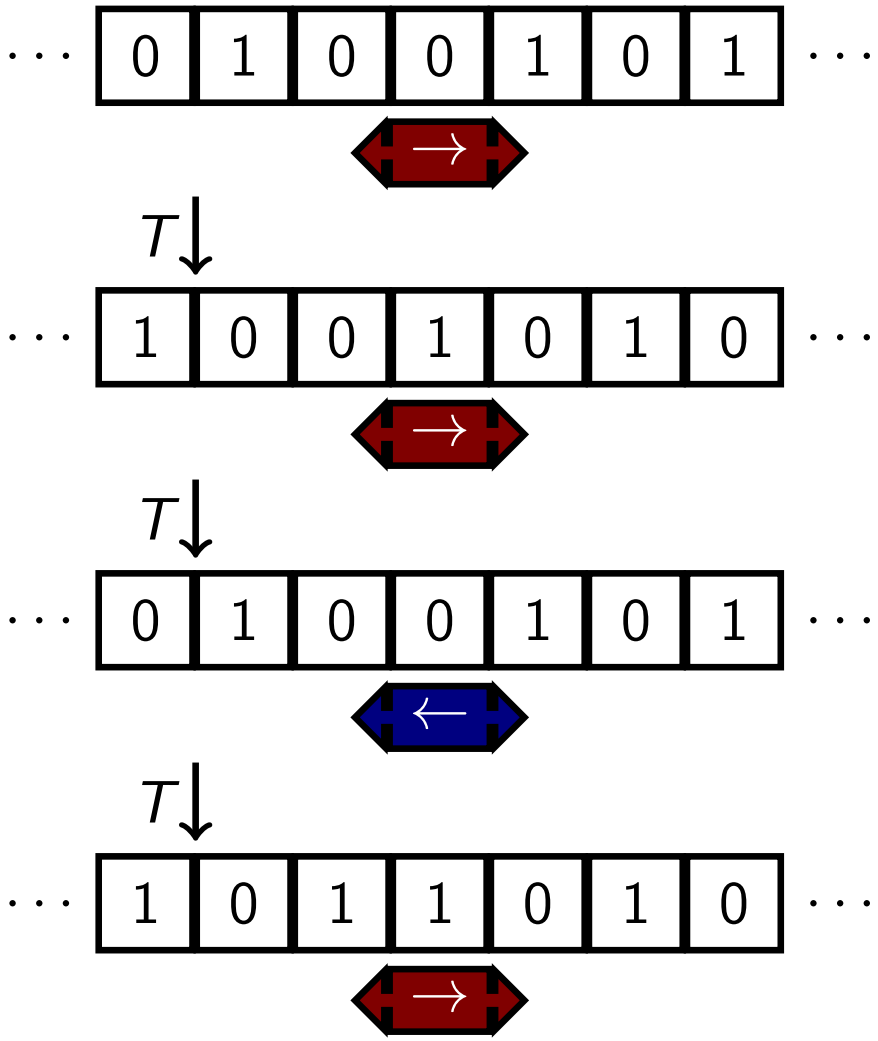


Column Factor of TMT (t-shift)



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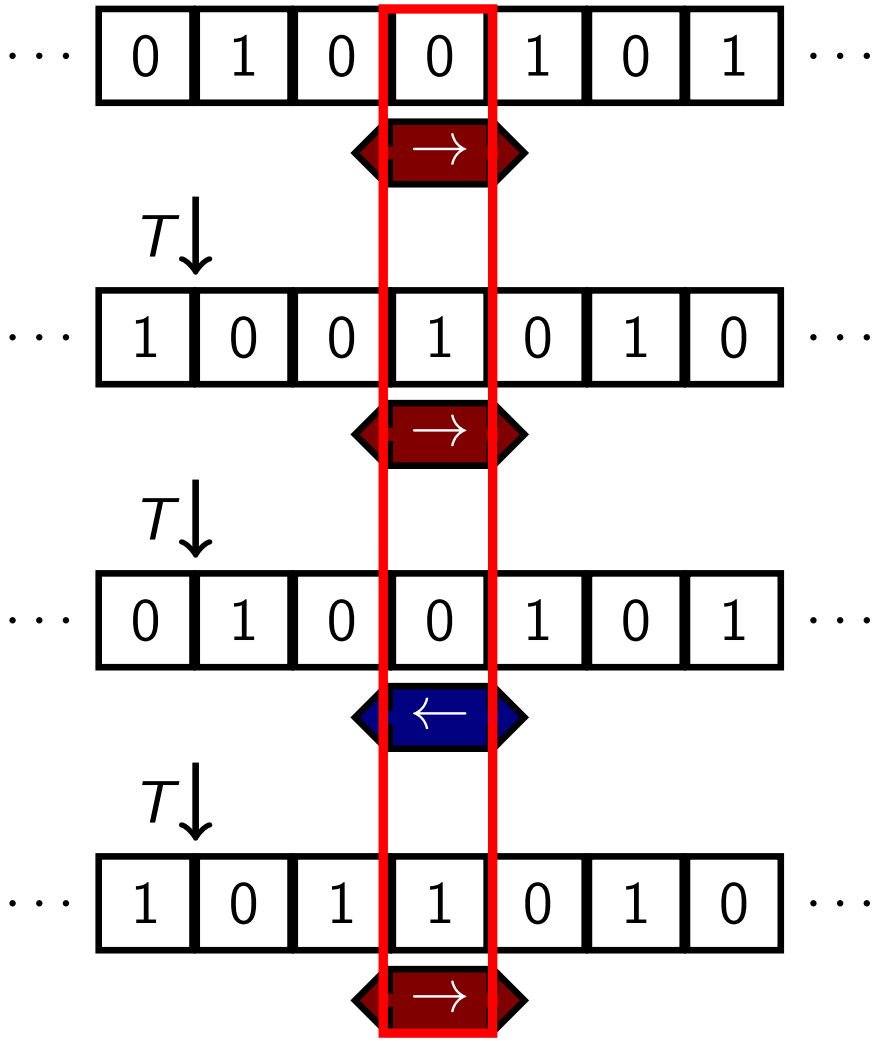

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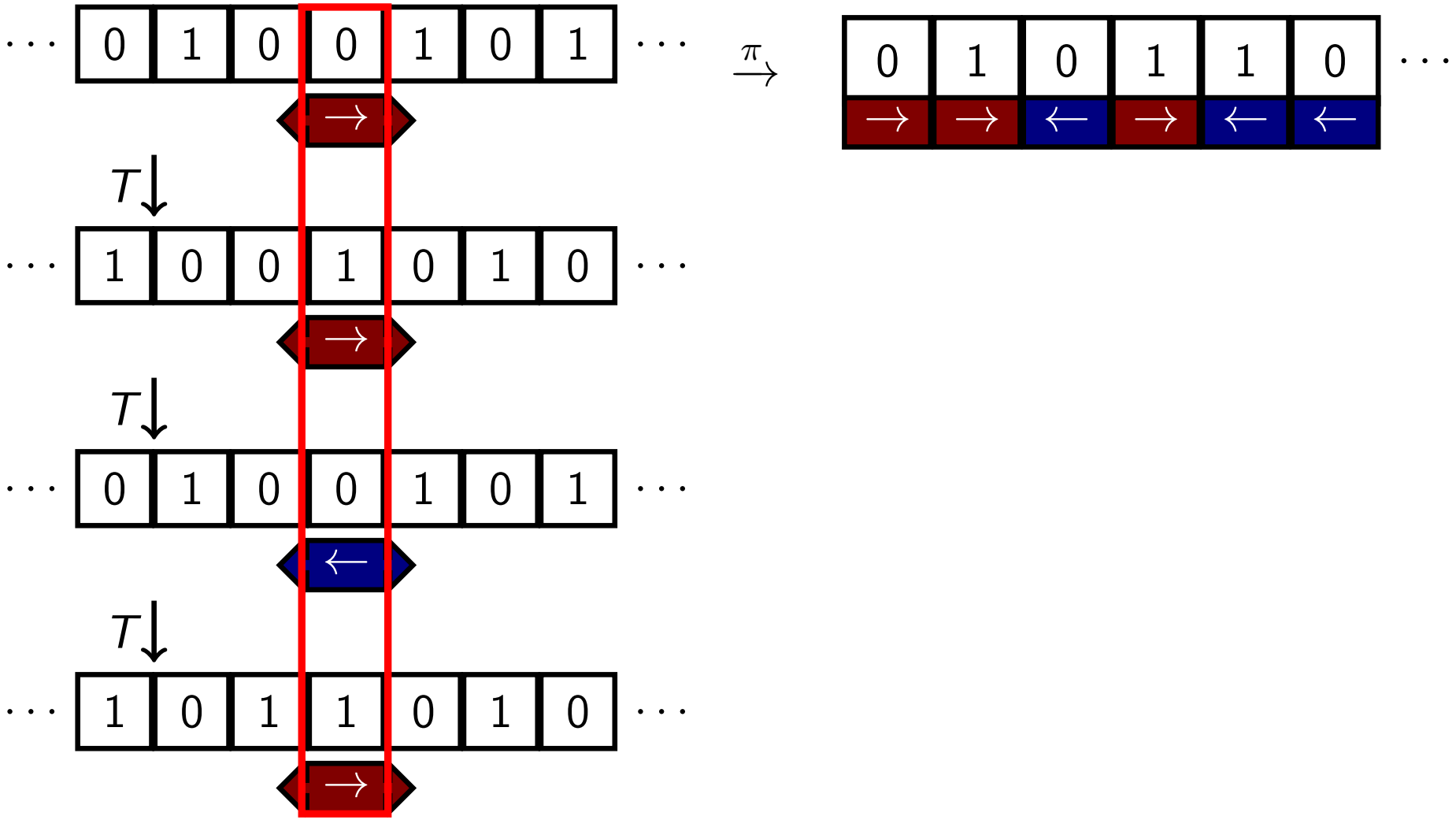

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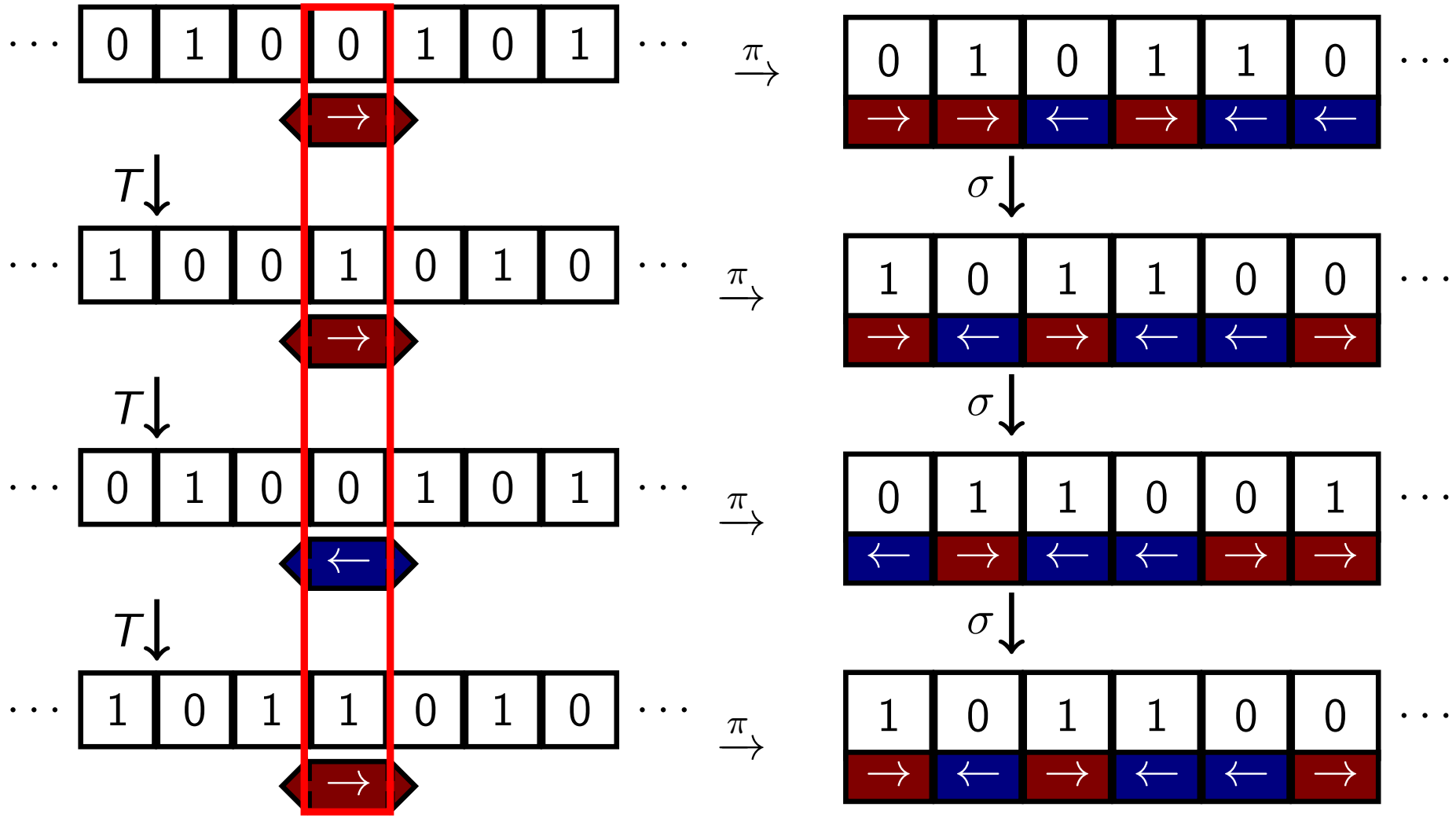

Interesting considerations:

- TMH and TMT [Kůrka '97] exists because $X = (Q, \mathbb{Z}, \Sigma^{\mathbb{Z}})$ is not compact, a serious draw back in order to be consider as topological systems.
- Both can be endowed with the cantor metric, being compact and perfect (no isolated points).
- Column factor of TMH can be consider, and also column factor of more than one cell (not in this talk).
- $\gamma : X \to X_t$ is onto and $\psi : X \to X_h$ is one-to-one.

$$\begin{tikzcd}
X_t \arrow[d, "T_t"] & X \arrow[l, "\gamma"'] \arrow[r, "\psi"] \arrow[d, "T"] & X_h \arrow[d, "T_h"] \\
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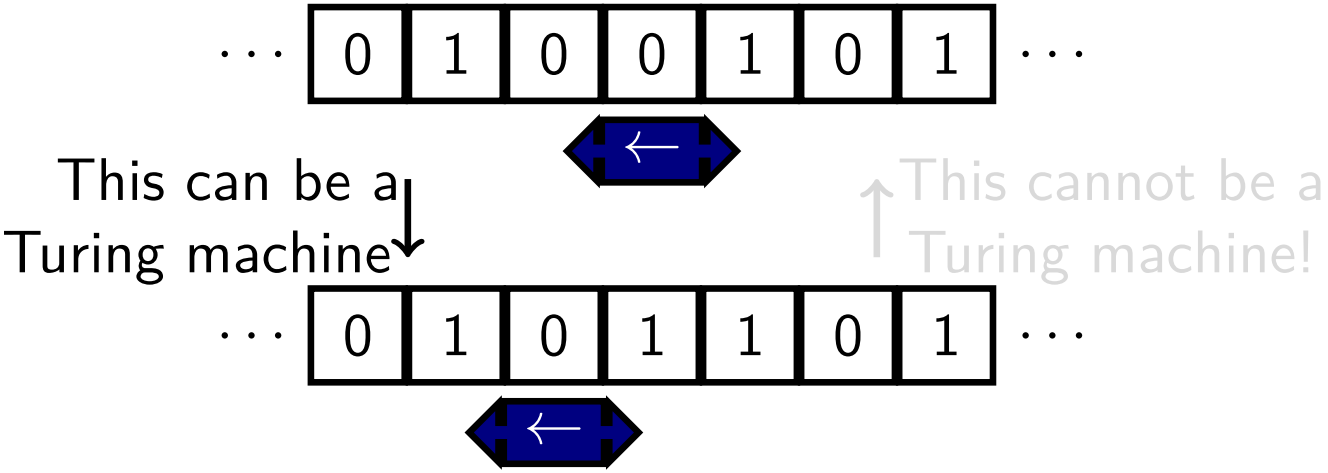
The problem with reversibility.



- Every state of a reversible Turing machine can just be attained from one direction. We consider $\mu : Q \to \{-1, 0, 1\}$.

- If $\varphi(w, i, r) = (w, i - \mu(r), r)$, then: $T^{-1} \circ \varphi \circ T = \varphi$

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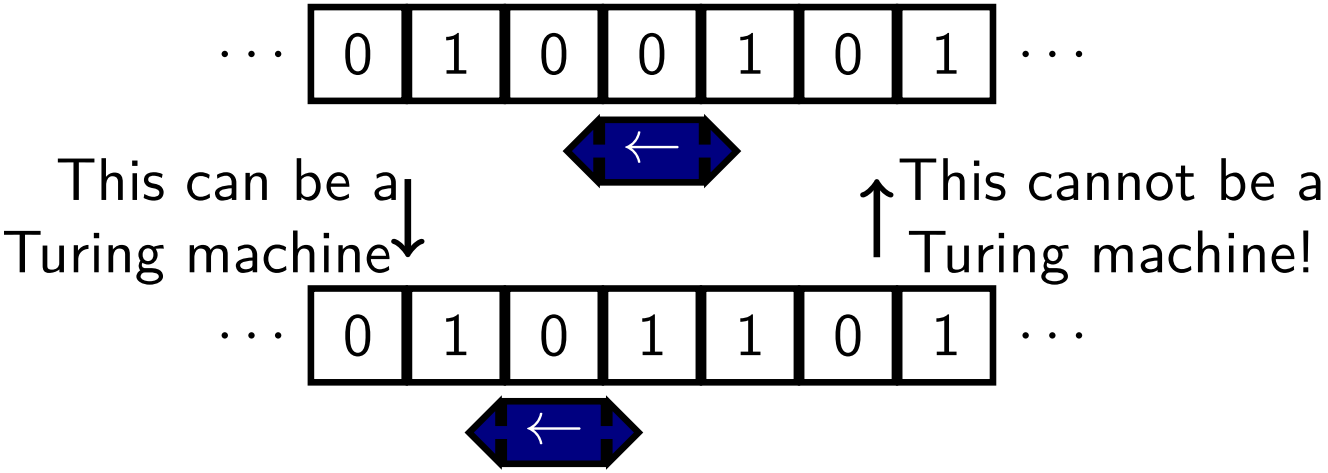

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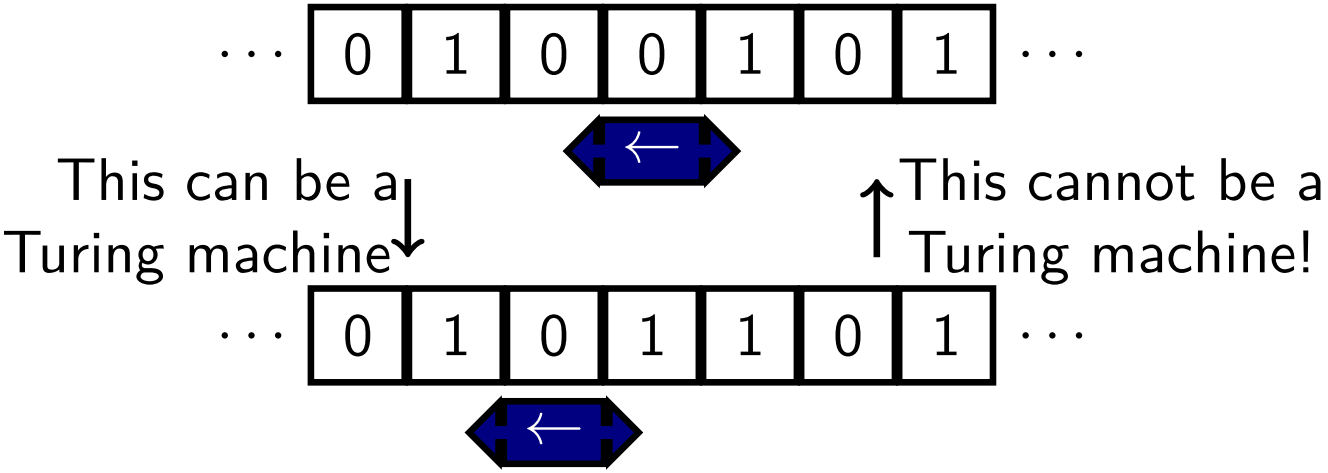

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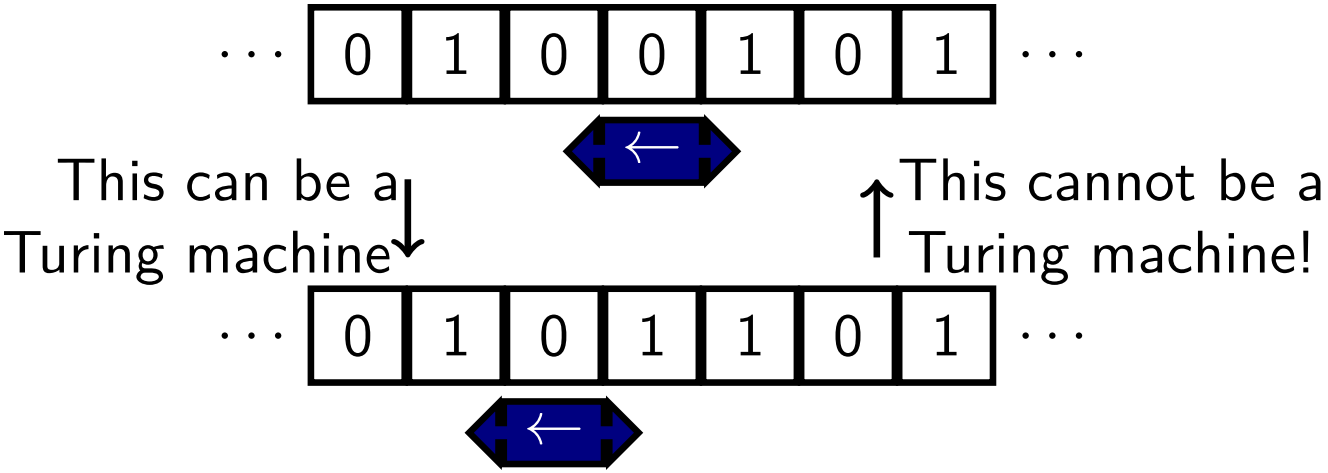

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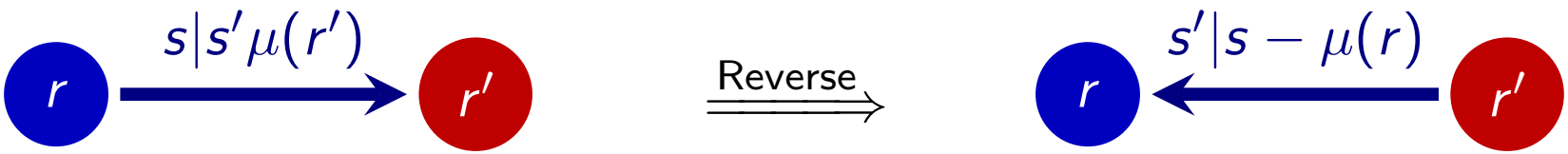

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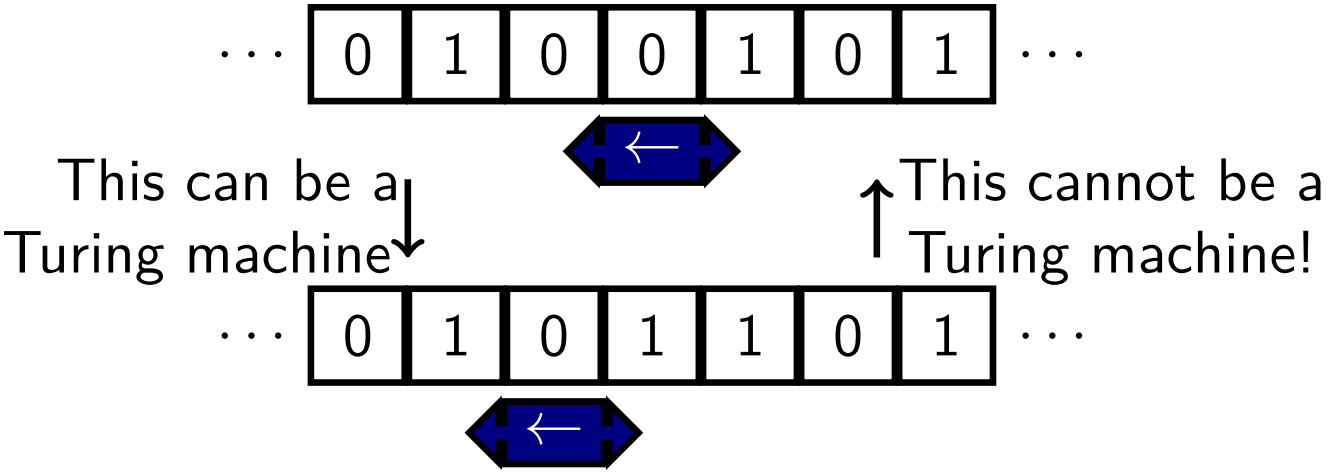

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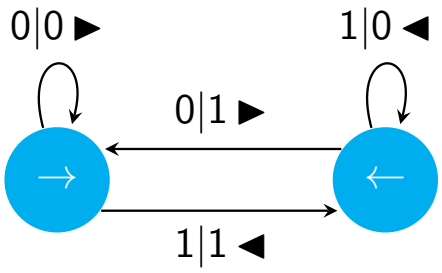
Another way to construct T^{-1}

1.-



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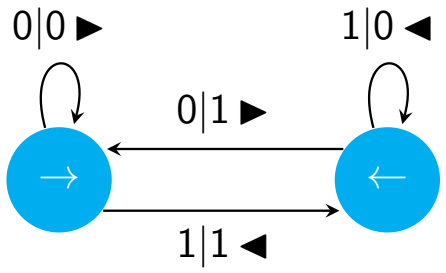
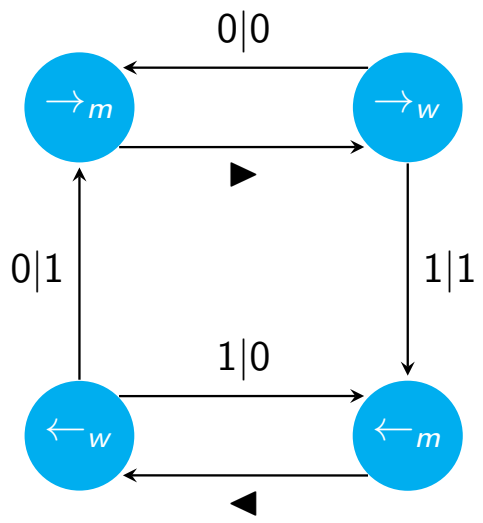


2.-



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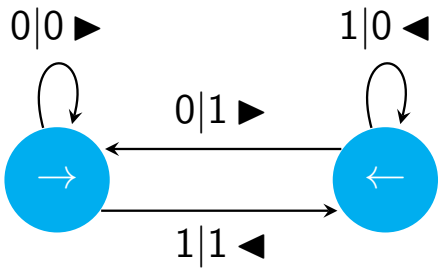

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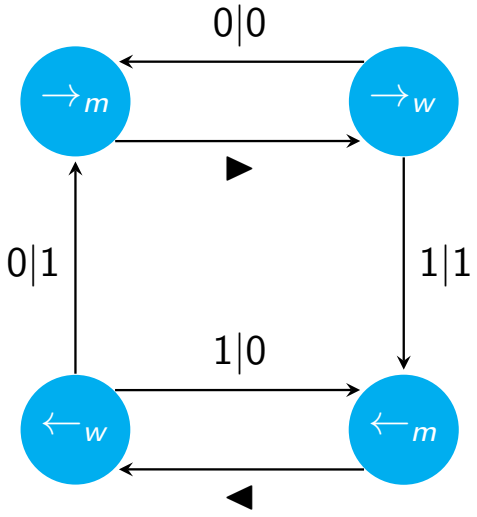

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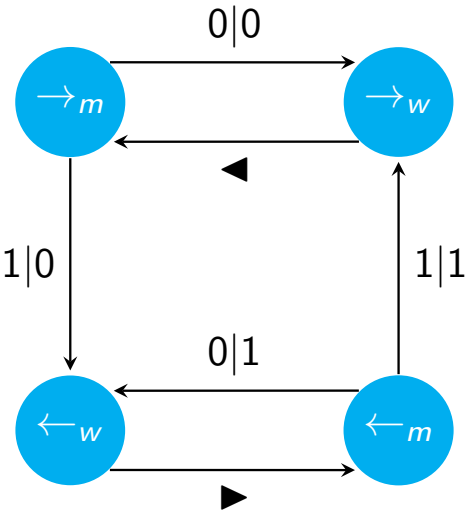

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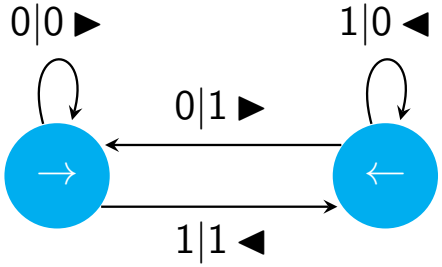

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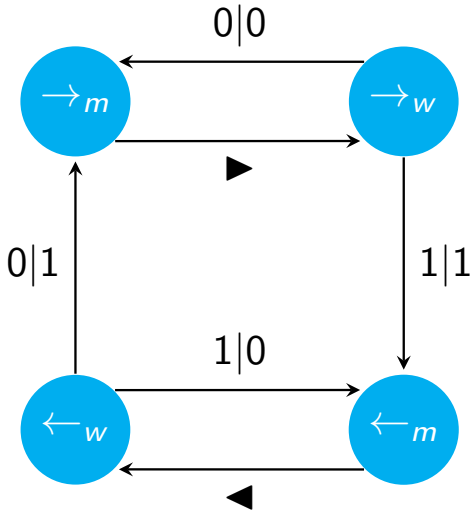

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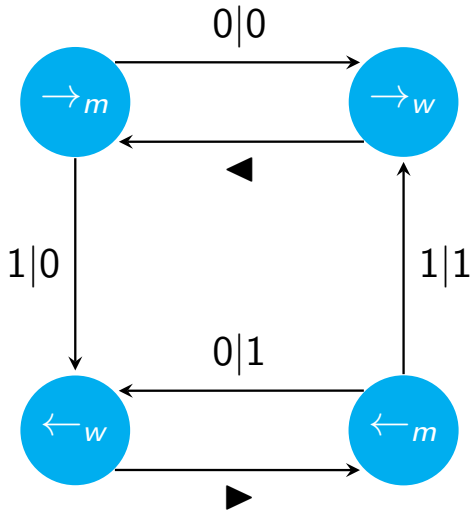

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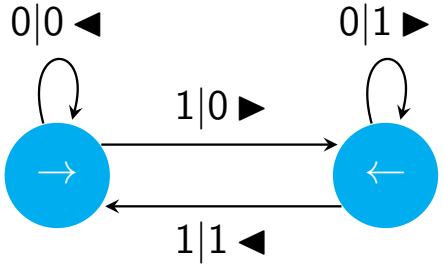
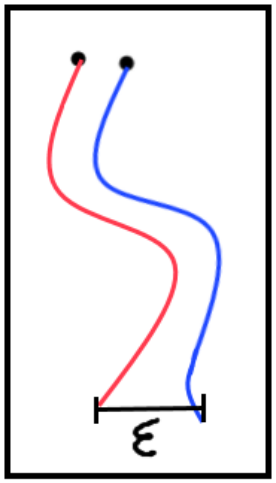
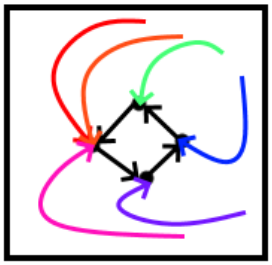


Content

Results in [Kůrka '97]

For every Turing machine, $\exists x \in X$ whose orbit never get the head to position -1. This roughly implies:





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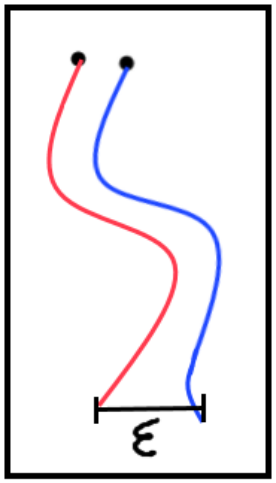

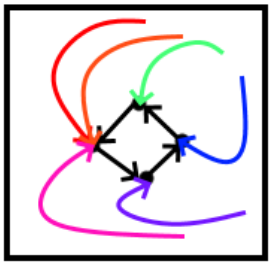

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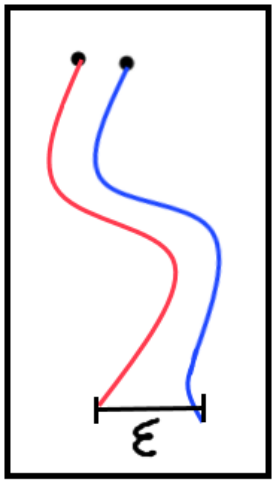

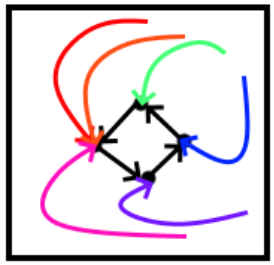

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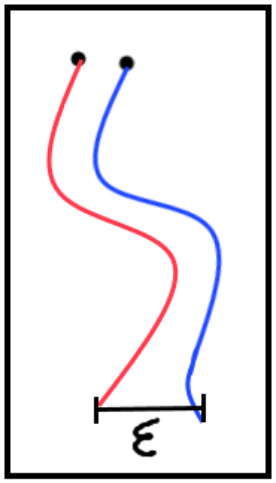

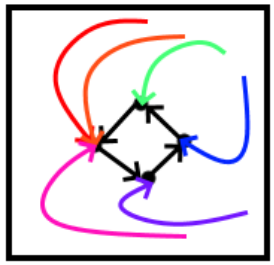

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More results in [Kůrka '97]

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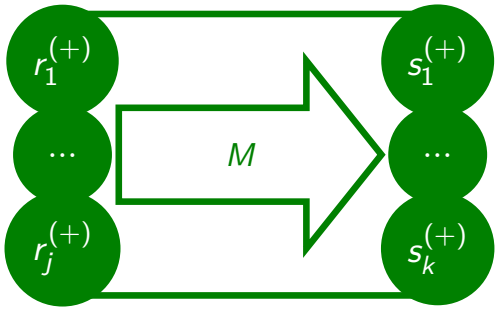
Content

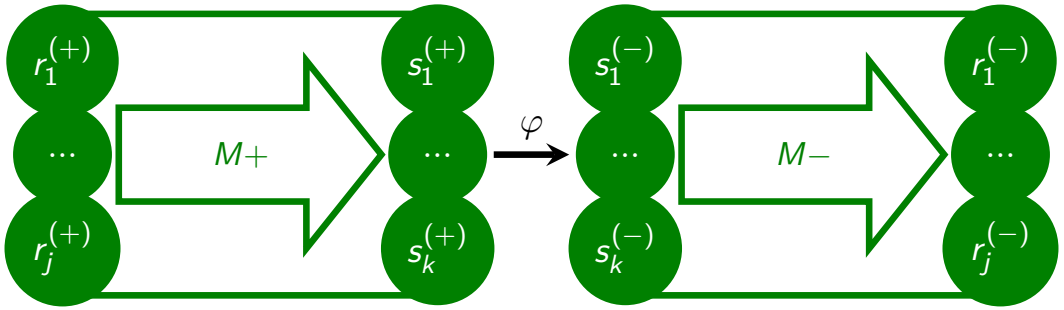
Useful technique to be used: Reversing the time

Consider incomplete and reversible TM M, with j defective pairs (state and symbol not in the image of δ), and k halting states.



Useful technique to be used: Reversing the time

We attach M^{-1}, connecting every $\delta(s_i, a) = \mathit{undef}$ with $\delta(s_i^{(+)}, a) = (s_i^{(-)}, a, -\mu(s_i))$



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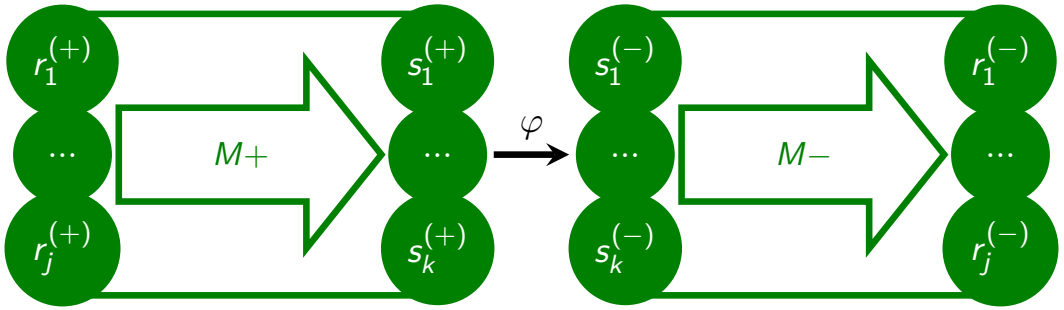

This machine, called I_M, has the property of being **Innocuous**: If the machine starts at $(r_l^{(+)}, i, w)$, it will halt at $(r_l^{(-)}, i - \mu(r_l), w)$, or compute indefinitely.

Context

- We center on TMT topological entropy.
- It is defined by its t-shift:

$$\mathcal{H}(T_t) = \lim_n \frac{\log |\mathcal{L}_n(S_T)^{\mathrm{i}}|}{n}$$

- An early work [Blondel et al., '04] showed the undecidability of topological entropy of TMT, but considering multiple tapes (based in product between Turing machines).

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- Jeandel [Jeandel '14] proved that topological entropy for one tape TMT can be computable with error $\epsilon > 0$.
- Using results from [Brudno '83]:

  $\exists x \in X : \mathcal{H}(T_t) = \lim_n \frac{K(\mathcal{L}_n(x))}{n}$[ii]
- It is proven that top. entropy is reached as a maximum in the set of configurations $c \in C^+ \subseteq X_t$ such that:
  - $\forall n > 0 : \rho(c, n) > 0$.
  - Every cell is visited a finite amount of time.

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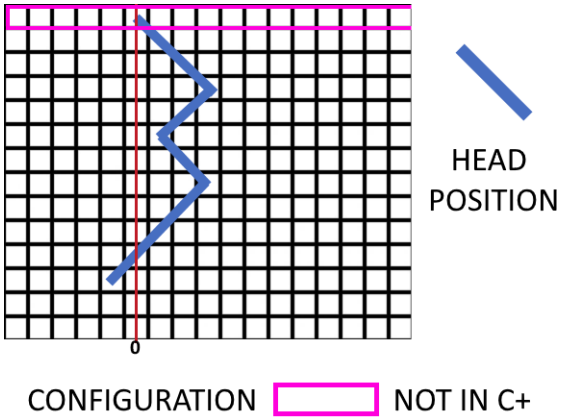

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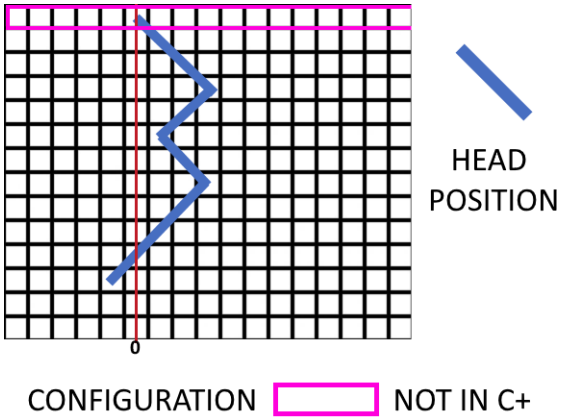

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How to compute the entropy

- To calculate entropy considering first n cells, it is constructed a Finite Automaton considering all possible paths in C^+ (finite) and calculate its (Kolgomorov) complexity.
- This finite automaton has the *diamond property*: given two vertices a, b and a word u, there is at most one path from a to b labeled by u.
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But then, is it possible to decide if $\mathcal{H}(T_t) = 0$?

Theorem (Gajardo, Ollinger, TA '15)

To decide if a (reversible and) complete Turing machine has a entropy of 0 *is undecidable.*

Proof Sketch

- A simple simulation of a 2 Reversible Counter Machine C in the right side of the tape is done:
  - Input $<_s 1^n|1^m>$ represents configuration (s, n, m) of C.
  - For a successful computation, we need enough work space (represented by symbols 1 in the tape).
- We distinguish 4 types of configurations:
  - Configurations with garbage in the tape.
  - Configurations with unbounded searches of symbols: $|, >, <_s$.
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- Configurations with garbage will generate periodic points, only seeing a finite amount of tape.
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- Infinite correct computations will visit cell 0 infinitely often.
- Therefore, the unique configuration with entropy greater than 0 exists if C with empty counters halts:

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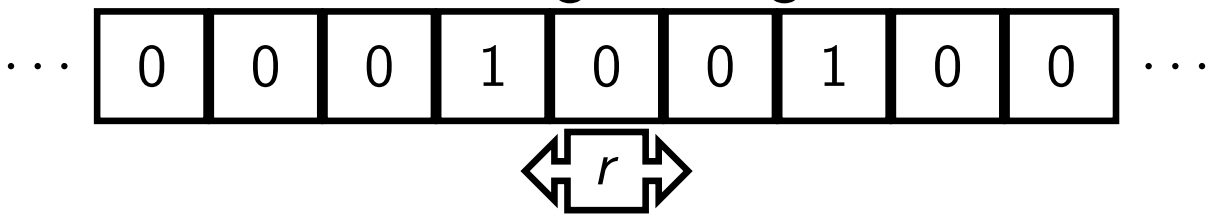

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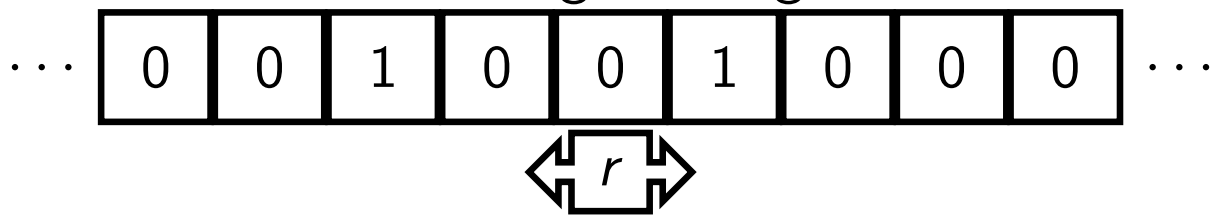

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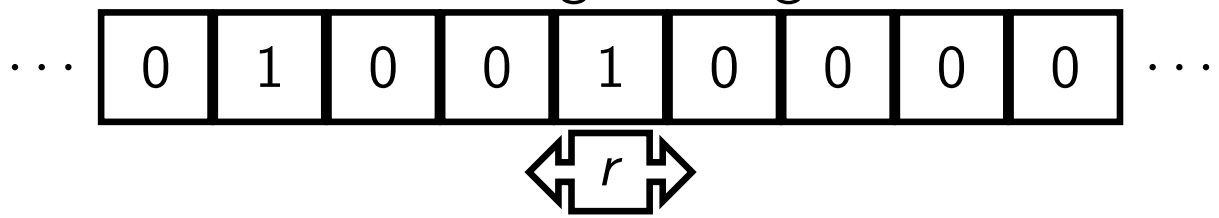

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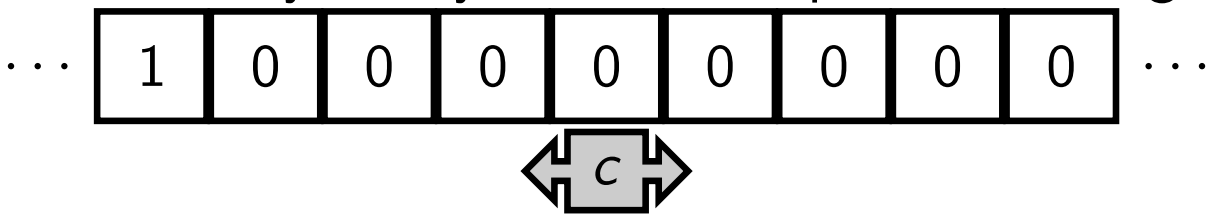
Idea: Carry the symbol 1 one space to the right.



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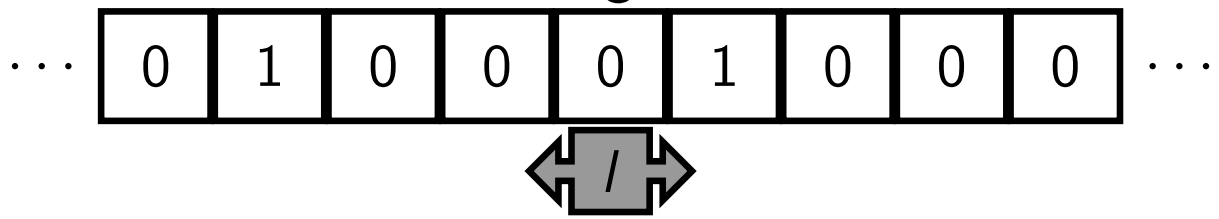
Idea: Returning to 1 at the left.



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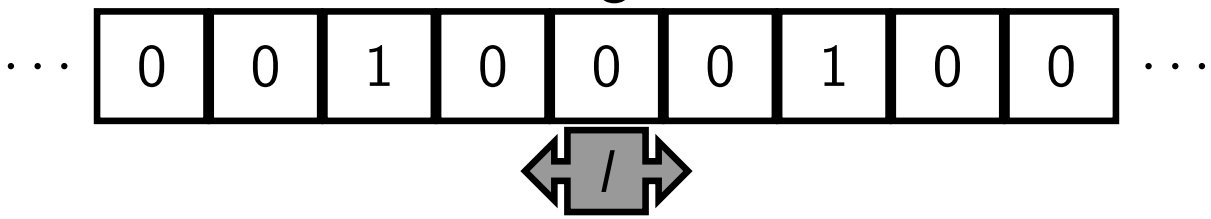

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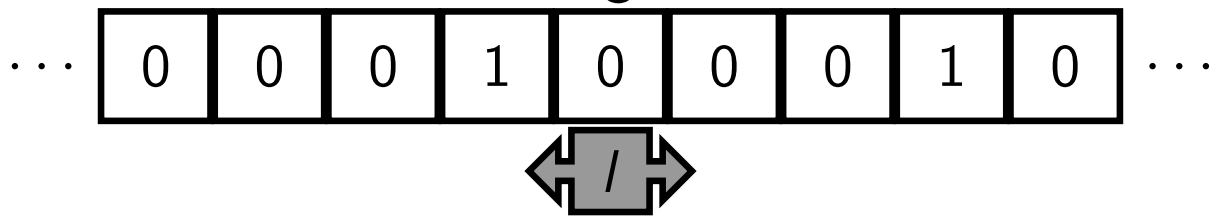

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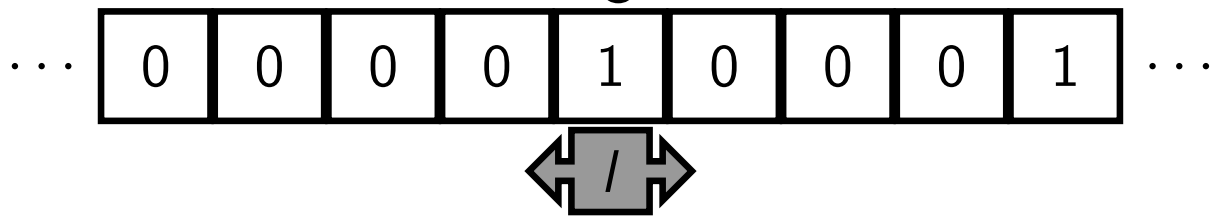

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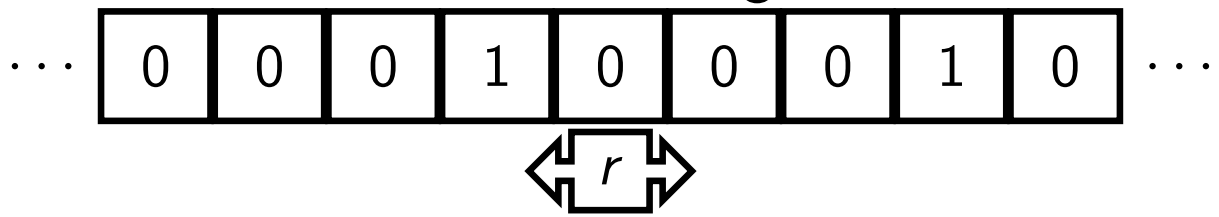
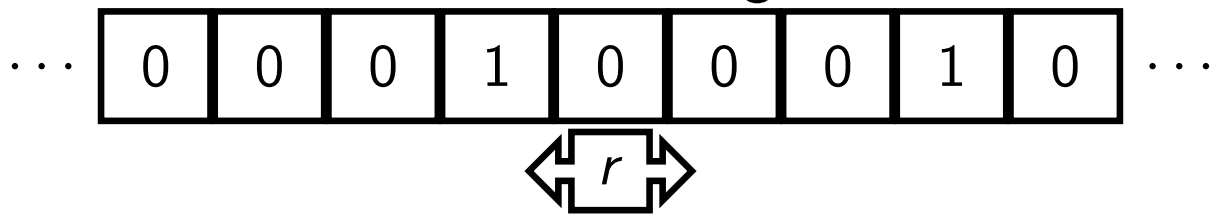
Idea: Start again.



How to aperiodic: Idea

Problem: If configuration has no 1 to the left or right.

Idea: Start again.



How to Aperiodic: Solution

Hopper Call! Roughly speaking, this make a bounded call inside the search space, pinging a 1 to the right, leaving a 1 in the original position.

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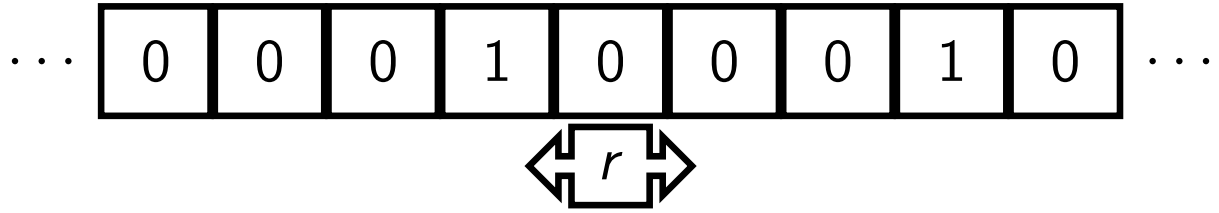

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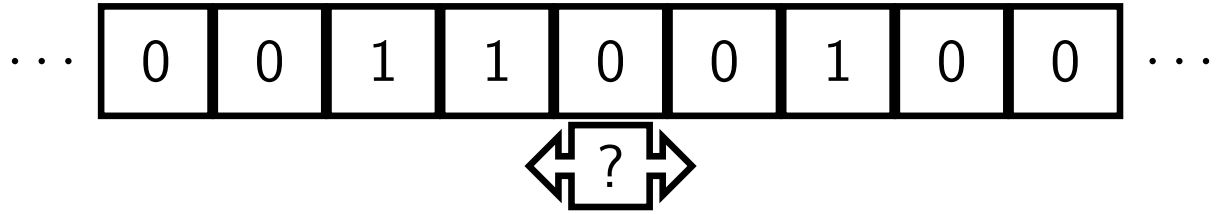

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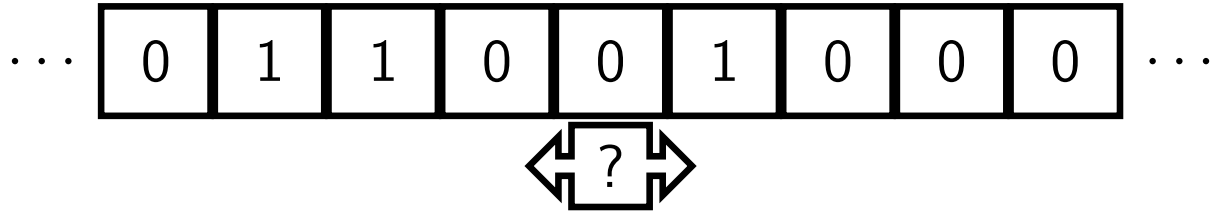

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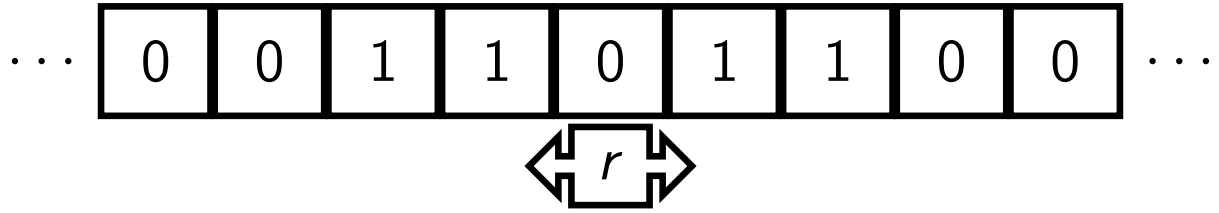

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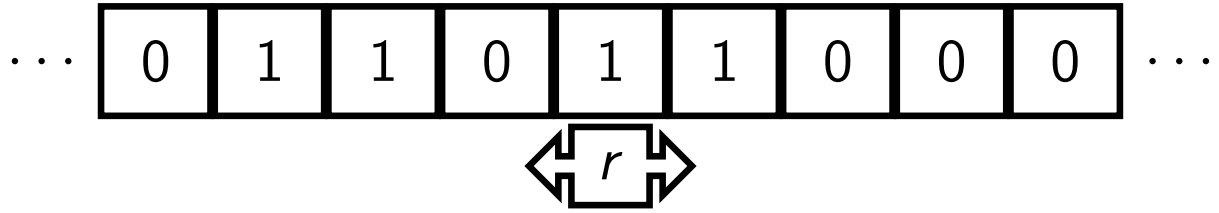

How to Aperiodic: Solution

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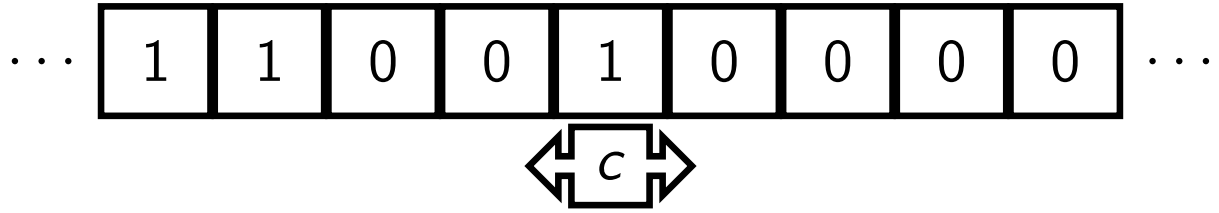
And when you have no more space...



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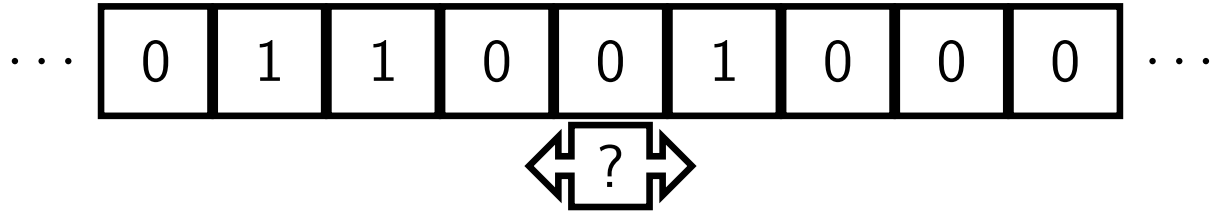

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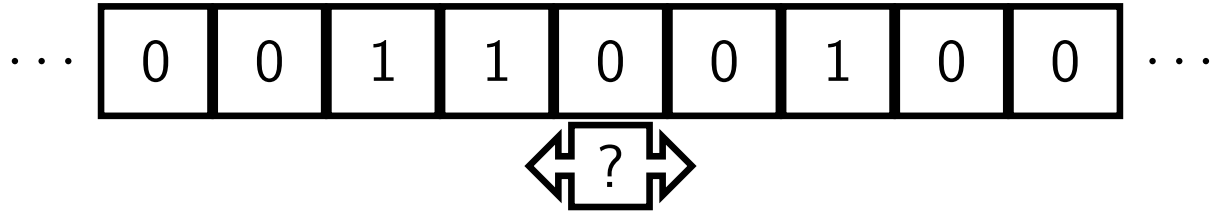

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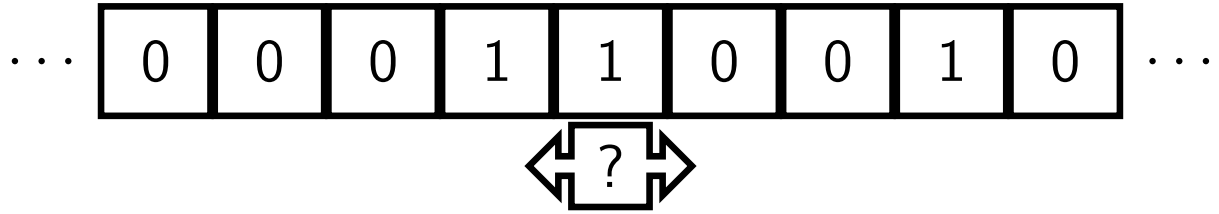

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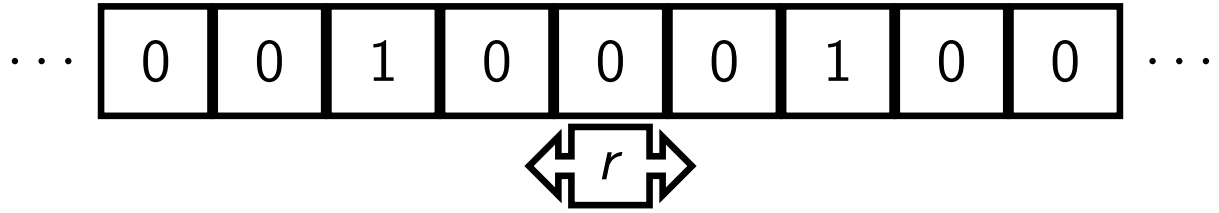

Hooper's Call

- It was introduced by [Hooper '66], to prove the undecidability of wheter a Turing machine was **Mortal**.
- An incomplete Turing machine is called **Mortal** if it halts in every possible configuration[iii].
- This study was further completed by [Kari, Ollinger '08], proving undecidability of mortality in reversible Turing machines, and later on by [Jeandel '12], by characterizing immortal configurations (which does not halt).

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Disproving Kůrka's conjecture

Theorem (Blondel, Cassaigne, Nichitu '02)

There exists a complete Turing machine aperiodic in TMT.

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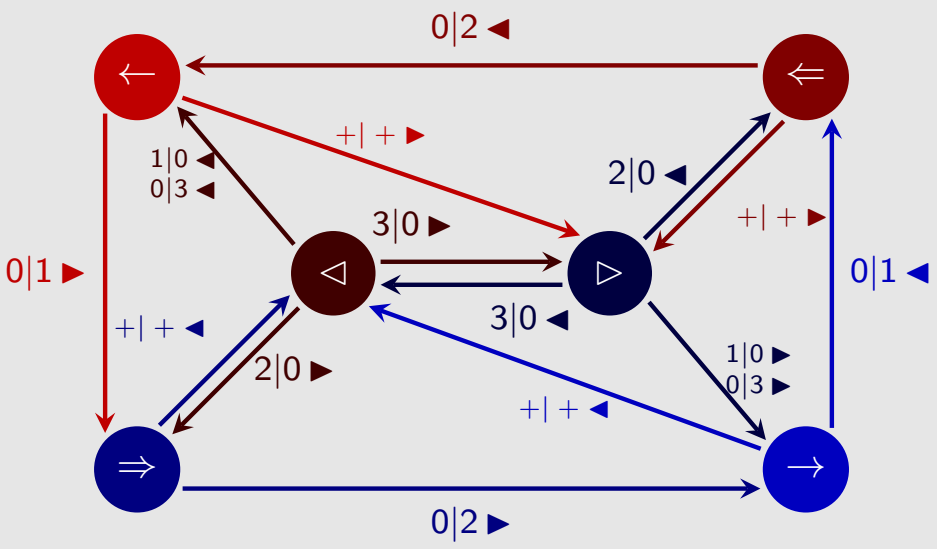

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- As can be seen in the diagram, the machine is not reversible.
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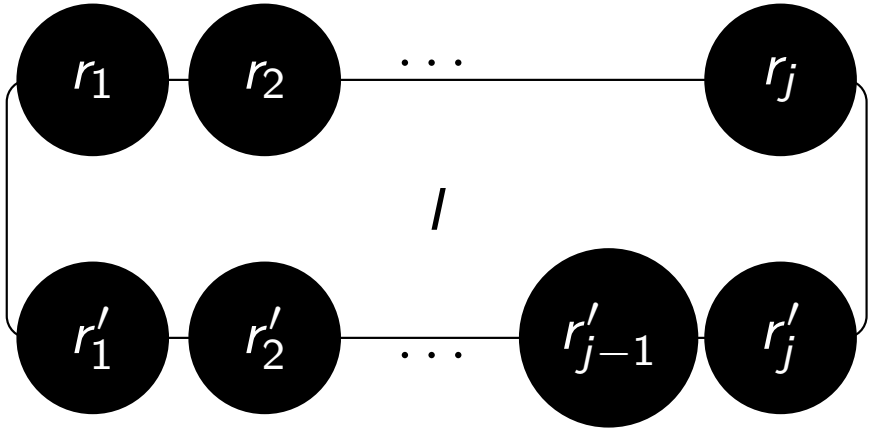
Useful technique to be used: Embedding



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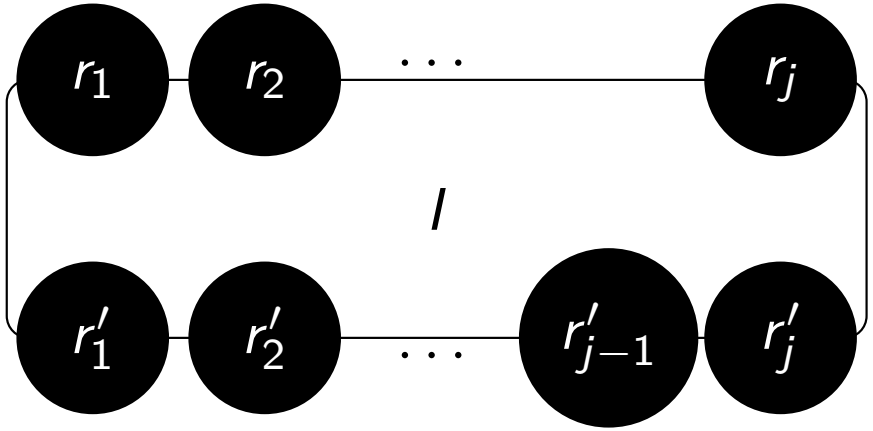

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- Machine I needs to be *innocuous*, which can be attained by using *reversing the time*.

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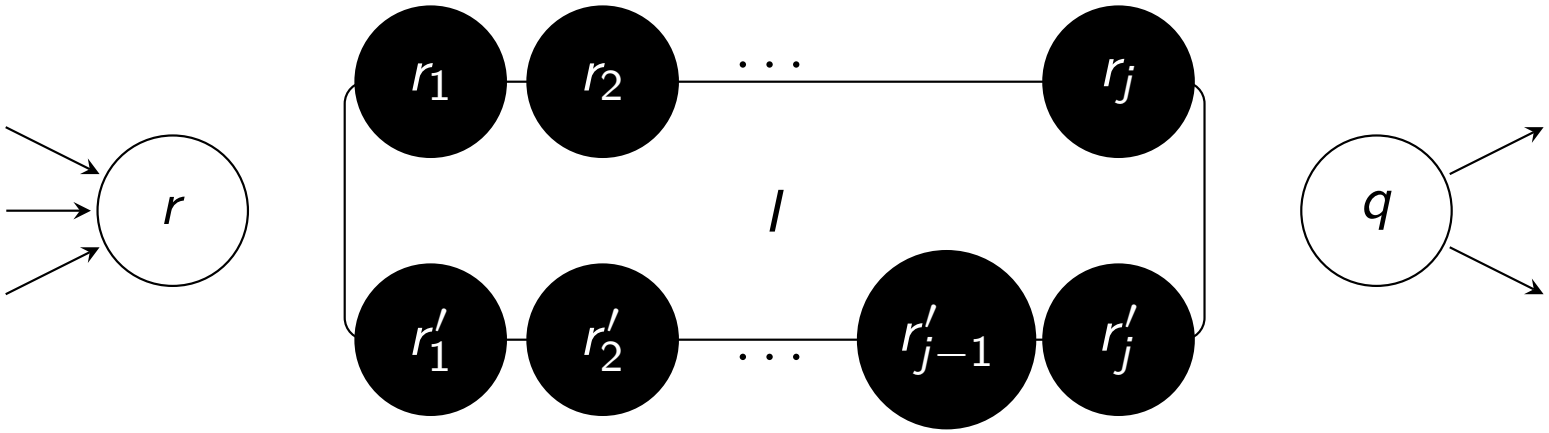

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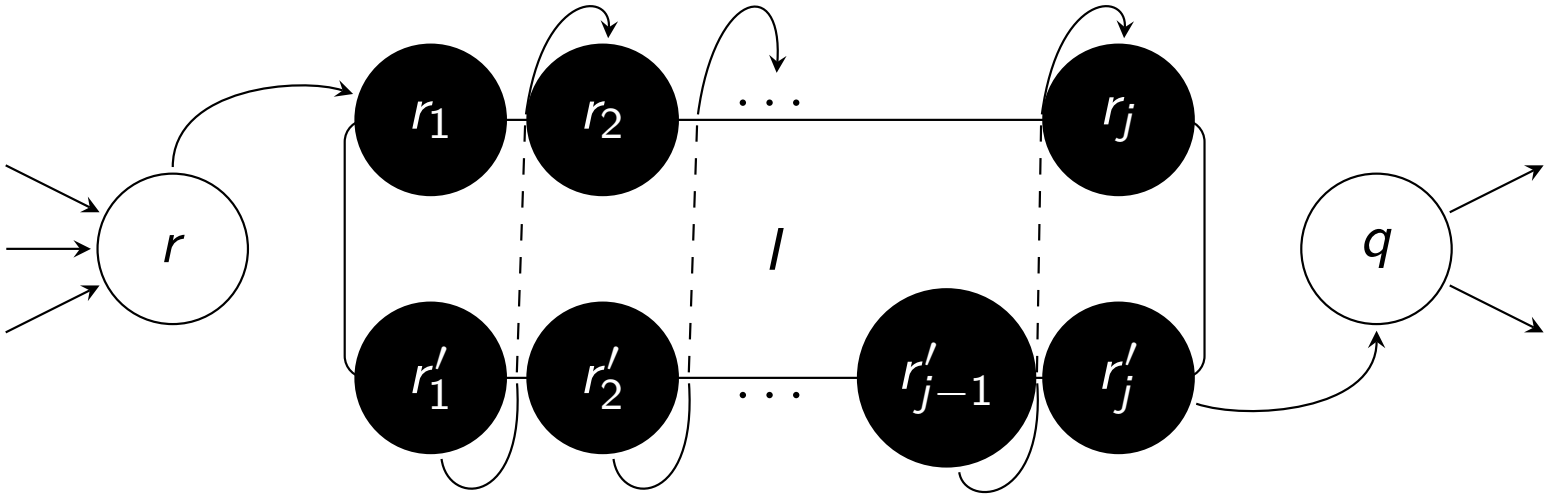

Proving Kari & Ollinger conjecture

Theorem (Cassaigne, Ollinger, TA '17)

There exists an aperiodic complete and reversible TM in TMT with three symbols. It is called SMART machine.

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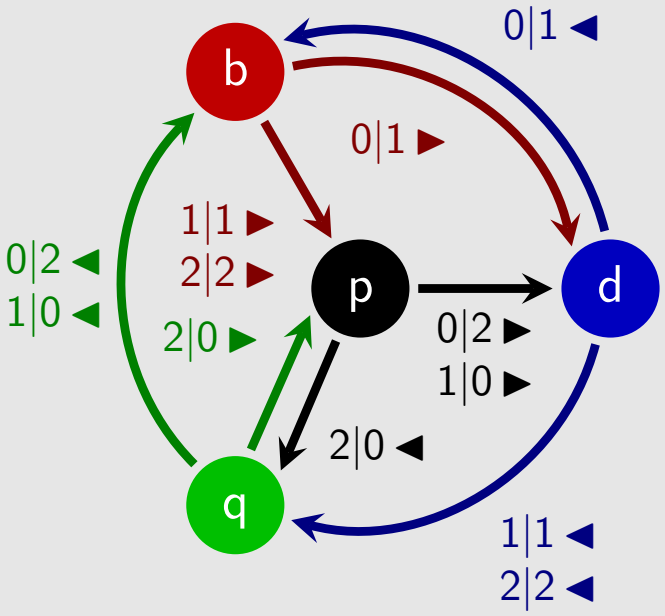

SMART machine

- This machine uses the same idea of Hooper's call pushing a positive symbol to the right. But this machine also does it to the left, in a recursive way.
- This machine is symmetric in the space and time.
- Also, its t-shift is also a **substitution**. (the closure of a replace from words to words).
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Topological Transitivity in TMH

For every $u,v \in (\Sigma \times Q)^* : |u|_Q \leq 1, |v|_Q \leq 1$

- Using *embedding*, it is possible to prove undecidability of this property with a reduction of the aperiodicity problem for reversible machines.

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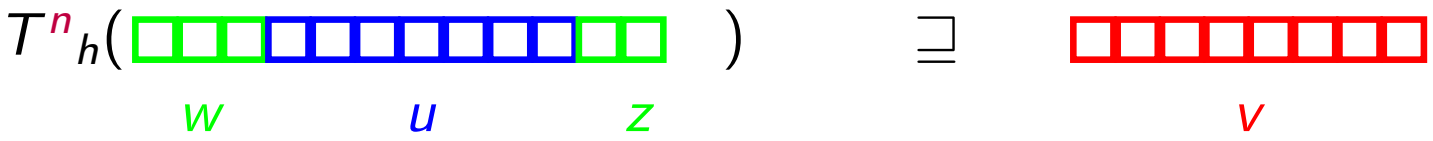
Exist $w, z \in (\Sigma \times Q)^*$ and $n \in \mathbb{N}$ such that:



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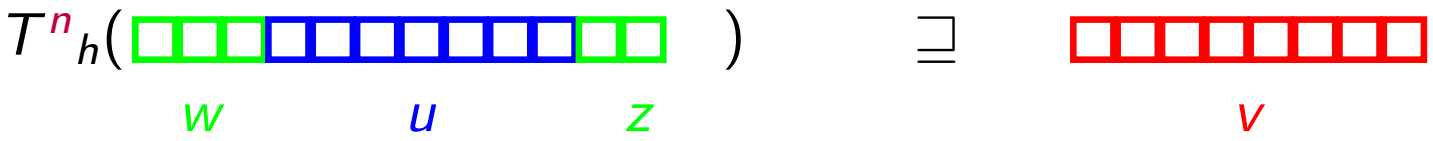

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Topological Minimality in TMT

For all $x \in X_t$ and $v \in \Sigma^* \times Q \times \Sigma^*$

- For minimality, we need no periodic points. Then, it is expected than our first minimal machine is discover within the search of aperiodic Turing machines.
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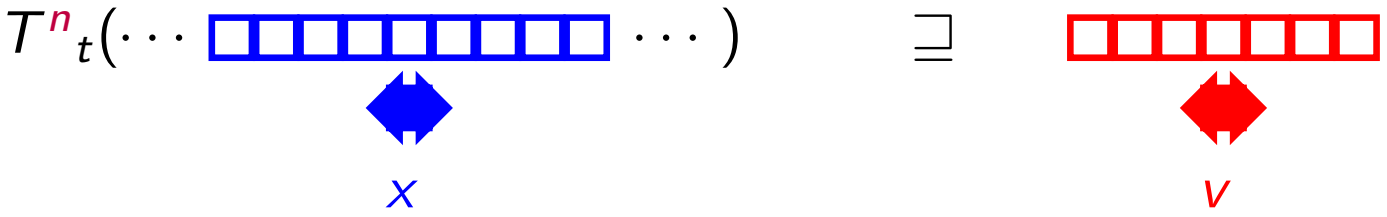

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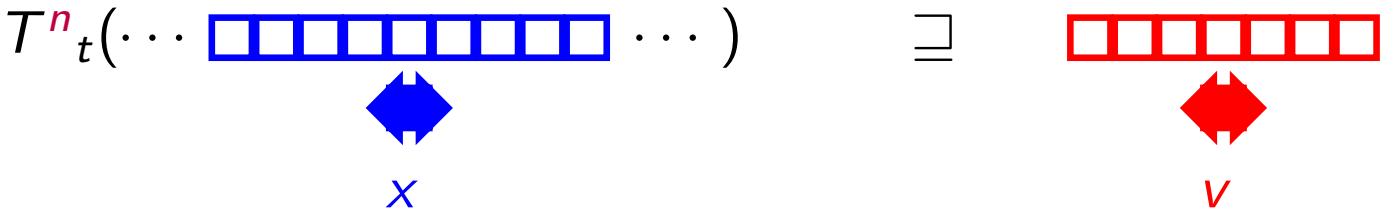

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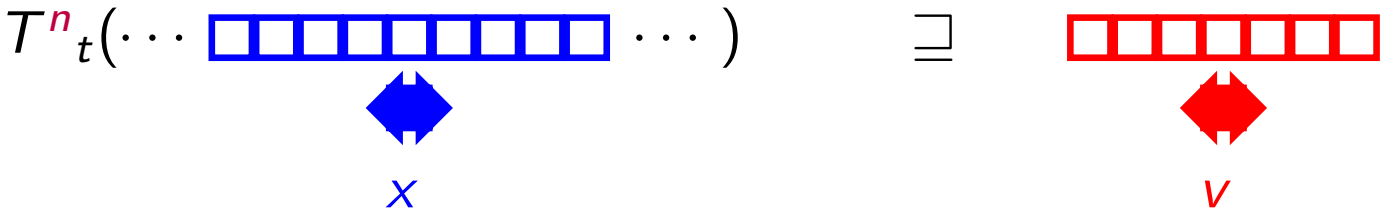

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The world of transitive t-shift

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- Blocking words [Gajardo, Ollinger, **TA** '12]:

$$(\exists(,r,u) \in (\Sigma^* \times Q \times \Sigma^*))(\forall w \in \Sigma^\omega)(\forall n \in \mathbb{N}) : \rho((,r,uw),n) \geq 0$$

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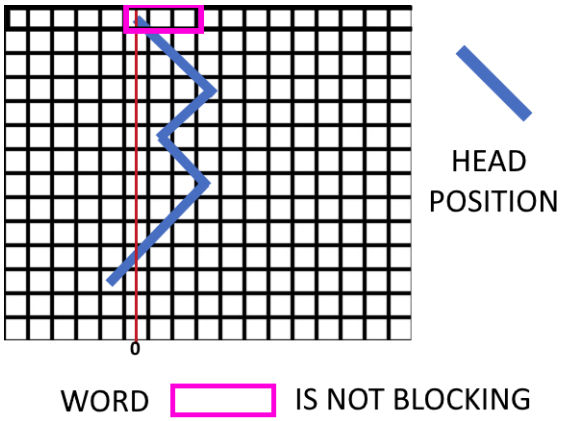

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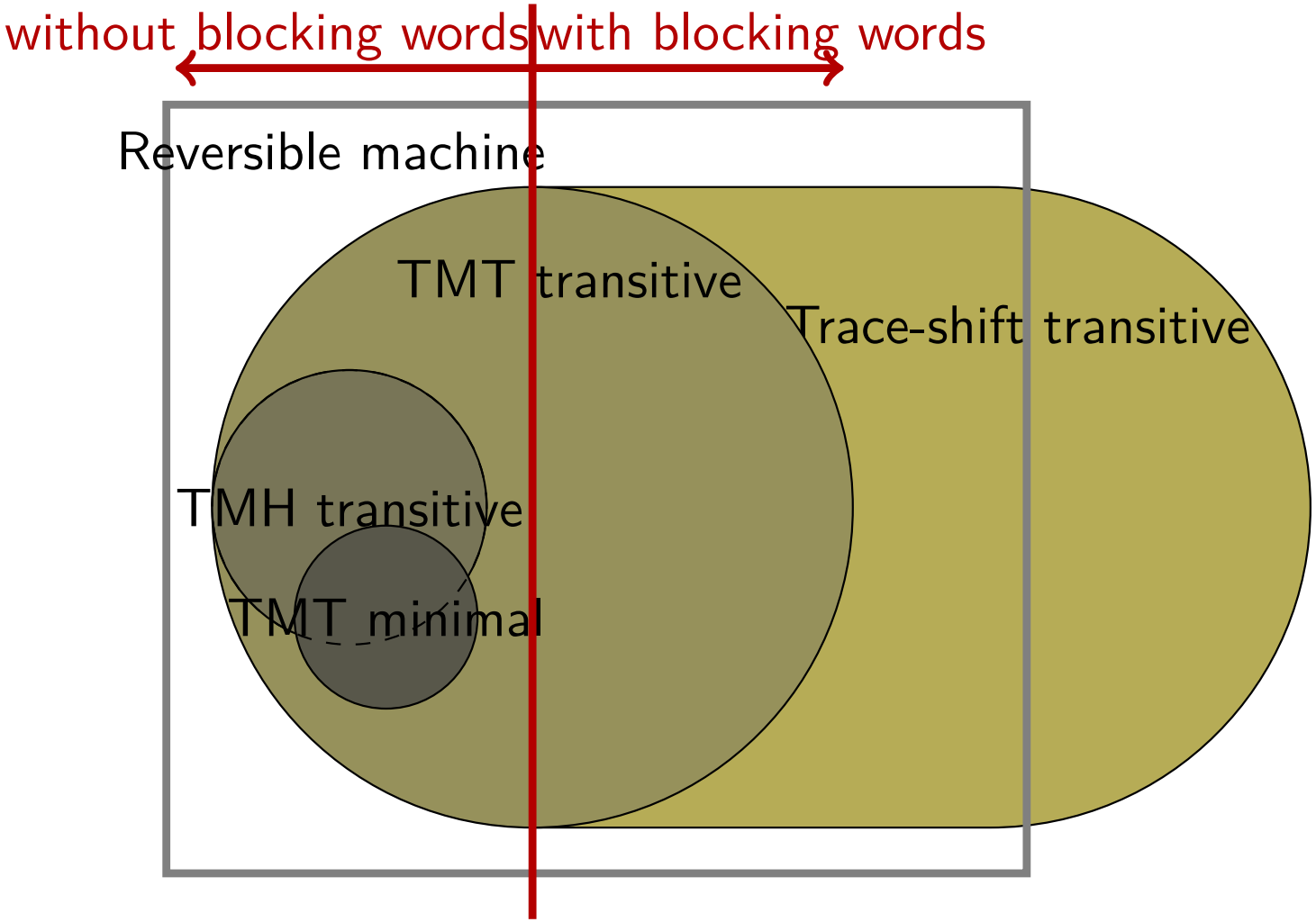

Embedding properties

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- Embedding technique is proven to kept some properties from the machines that formed it. Consider H the complete machine, I the innocuous machine, and H_I the formed machine. Then:
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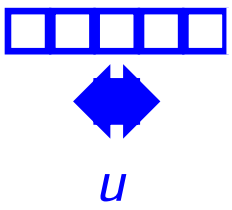
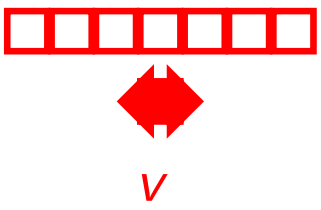
$\forall u, v \in \Sigma^* \times Q \times \Sigma^*$ and $\forall N \geq n$ (which depends in u and v)

$T^N{}_t($  $)$ 

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- Considering *embedding*, it is possible to prove that **Every mixing notion is undecidable for Turing machines**, with a reduction of the aperiodicity problem.
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- Much of the work adjacent to the topological properties were not touch, as the interesting work of Guillon, Salo, Gajardo and many others (apologies in advance).
- Gajardo et al. have results considering Sensitivity, Equicontinuity and Soficity.
- An interesting result of [Guillon, Salo '17] is that aperiodic machines as a sub-linear movement in the space.

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Open questions

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- Direct questions arise as: There exists a minimal non-linearly recurrent Turing machine?
  - In linearly recurrent systems, word u can be seen in the trace again at a maximum of $K * |u|$ space, with K depending on the system.
  - If it would not exists such a Turing machine, Topological Weak-Mixing is decidable in Minimal Turing machines.
- Other interesting questions is about the arithmetical hierarchy of some topological properties (which are also related to the previous question).
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